ECON 720: Advanced Macroeconomic Theory I

Continuous-Time Methods

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Overview

- So far: discrete time, $t = 0, 1, 2, \ldots$. Now: continuous time, $t \in [0, \infty)$
- Why bother?
  - derivatives, not differences: growth rates add up, no timing conventions
  - dynamics are ODEs: phase diagrams, often closed forms
  - the workhorse in growth, search, and heterogeneous-agent macro
- Plan
  1. The Solow model in continuous time: notation and ODEs
  2. From the Bellman equation to the **HJB equation**
  3. Optimal control: why the HJB (usually) works; Tobin's q
  4. Uncertainty: Poisson jumps in the HJB
- One trick throughout: period length Δ in discrete time, then $\Delta \to 0$

The Solow Model in Continuous Time

Notation

- Time $t \in \mathbb{R}_+$. For any variable $x(t)$, its time derivative and growth rate:

$$\dot{x}(t) := \frac{dx(t)}{dt}, \qquad \frac{\dot{x}(t)}{x(t)} = \frac{d \ln x(t)}{dt}$$

- Constant growth: $\dot{x}(t) = gx(t) \iff x(t) = x(0)e^{gt}$
- **Stocks vs. flows**
  - a stock is measured at an instant: capital $K(t)$
  - a flow is a rate per unit of time: output $Y(t)$, consumption $C(t)$, investment $I(t)$. Over a short interval of length Δ, output is $\approx Y(t)\Delta$
- Growth rates of products and ratios add up exactly (log additive): if $k = K/(AL)$, then

$$\ln k = \ln K - \ln A - \ln L \quad \Longrightarrow \quad \frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L}$$

Capital accumulation as $\Delta \to 0$

- δ is a depreciation *rate*: per unit of time. Over an interval of length Δ, a fraction $\delta\Delta$ of capital wears out, and investment adds $I(t)\Delta$:

$$K(t+\Delta) = (1-\delta\Delta)K(t) + I(t)\Delta \quad \iff \quad K(t+\Delta) = K(t) + \big[I(t) - \delta K(t)\big]\Delta$$

- Rearrange and send $\Delta \to 0$:

$$\frac{K(t+\Delta) - K(t)}{\Delta} = I(t) - \delta K(t) \quad \xrightarrow{\Delta \to 0} \quad \dot{K}(t) = I(t) - \delta K(t)$$

- With $\Delta = 1$, this is the familiar $K_{t+1} = (1-\delta)K_t + I_t$

The fundamental equation

- Solow: $Y = F(K, AL)$ with CRS and the usual assumptions, $I = sY$, $\dot{L} = nL$, $\dot{A} = \gamma A$
- Let $\widetilde{k} := K/(AL)$ and $f(\widetilde{k}) := F(\widetilde{k}, 1)$. Since $\frac{\dot{\widetilde{k}}}{\widetilde{k}} = \frac{\dot{K}}{K} - \gamma - n$ and $\frac{\dot{K}}{K} = \frac{sY}{K} - \delta$,

$$\dot{\widetilde{k}} = \underbrace{sf(\widetilde{k})}_{\text{inflow}} - \underbrace{(\delta + \gamma + n)\widetilde{k}}_{\text{outflow}}$$

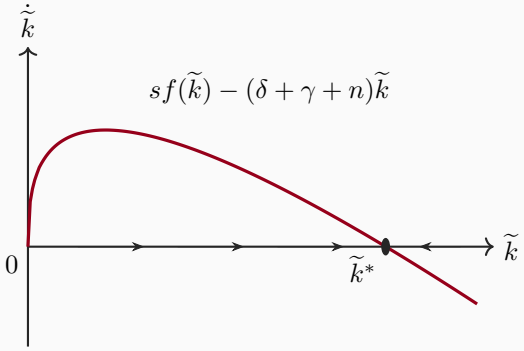
  a first-order ODE, given $\widetilde{k}(0)$. Inflow: investment per effective worker. Outflow: depreciation, plus dilution as AL grows

- Compare discrete time:

$$(1+\gamma)(1+n)\widetilde{k}' = sf(\widetilde{k}) + (1-\delta)\widetilde{k} \quad \Longleftrightarrow \quad \widetilde{k}' - \widetilde{k} = \frac{sf(\widetilde{k}) - (\delta + \gamma + n + \gamma n)\widetilde{k}}{(1+\gamma)(1+n)}$$

  In continuous time, the cross term γn and the denominator disappear

Steady state and global stability



- Steady state: $\dot{\widetilde{k}} = 0$

$$\frac{sf(\widetilde{k}^*)}{\widetilde{k}^*} = \delta + \gamma + n$$

- Unique $\widetilde{k}^* > 0$, and convergence is global and monotone: $\dot{\widetilde{k}} > 0$ below $\widetilde{k}^*$, < 0 above
- A **phase line**: one ODE, one dimension. With two ODEs, we will draw a phase *diagram*
- $\widetilde{k} = \widetilde{k}^*$: K, Y, C grow at rate $\gamma + n$, per-capita variables at γ (balanced growth path)

Cobb-Douglas: a closed-form path

- A useful fact: the **linear ODE** $\dot{y} = a - by$, $b > 0$, has the solution

$$y(t) = \frac{a}{b} + \Big(y(0) - \frac{a}{b}\Big)e^{-bt}$$

- With $f(\widetilde{k}) = \widetilde{k}^{\alpha}$, let $\kappa := \widetilde{k}^{1-\alpha}$. Then $\dot{\kappa} = (1-\alpha)\widetilde{k}^{-\alpha}\dot{\widetilde{k}}$, so

$$\dot{\kappa} = (1-\alpha)\big[s - (\delta + \gamma + n)\kappa\big]$$

  a linear ODE in κ. Hence

$$\widetilde{k}(t) = \left[\frac{s}{\delta+\gamma+n} + \Big(\widetilde{k}(0)^{1-\alpha} - \frac{s}{\delta+\gamma+n}\Big)e^{-(1-\alpha)(\delta+\gamma+n)t}\right]^{\frac{1}{1-\alpha}}$$

- The gap in $\widetilde{k}^{1-\alpha}$ (and, near $\widetilde{k}^*$, in $\ln\widetilde{k}$) closes at rate $(1-\alpha)(\delta+\gamma+n)$, the **speed of convergence**. E.g., $\alpha = 1/3$ and $\delta+\gamma+n = 0.06$: about 4% a year, a half-life of $\ln 2/0.04 \approx 17$ years

From the Bellman Equation to the HJB Equation

The planner's problem: discrete vs. continuous time

Neoclassical growth, no population or technology growth

	Discrete time, period length Δ	**Continuous time, $\Delta \to 0$**
time	$t = 0, \Delta, 2\Delta, \ldots$	$t \in [0, \infty)$
discount	$\beta_\Delta = e^{-\rho\Delta}$ per period, so $\beta_\Delta^{t/\Delta} = e^{-\rho t}$	$e^{-\rho t}$; $\rho > 0$ is the **discount rate**
objective	$\sum_{j=0}^{\infty} e^{-\rho j\Delta} u(c_{j\Delta})\Delta$	$\int_0^\infty e^{-\rho t} u(c(t))\, dt$
capital	$k_{t+\Delta} = k_t + \big[f(k_t) - \delta k_t - c_t\big]\Delta$	$\dot{k}(t) = f(k(t)) - \delta k(t) - c(t)$
bounds	$c_t, k_t \geq 0$, given $k_0 > 0$	$c(t), k(t) \geq 0$, given $k(0) > 0$

Useful limit: $(1 - \beta_\Delta)/\Delta = (1 - e^{-\rho\Delta})/\Delta \to \rho$. With $\Delta = 1$: $\beta = e^{-\rho} \approx 1 - \rho$ ($\rho = 0.04$ is $\beta \approx 0.96$)

Recursive formulation: Bellman vs. HJB

	Bellman, period length Δ	**HJB, $\Delta \to 0$**
state	k	k
control	$c \in \big[0,\ f(k) - \delta k + k/\Delta\big]$ (upper bound from $k' \geq 0$)	$c \geq 0$ (the upper bound $\to \infty$)
next state	$k' = k + \big[f(k) - \delta k - c\big]\Delta$	$\dot{k} = f(k) - \delta k - c$
equation	$V_\Delta(k) = \max_c \left\{u(c)\Delta + e^{-\rho\Delta} V_\Delta(k')\right\}$	$\rho V(k) = \max_{c \geq 0} \left\{u(c) + V'(k)\dot{k}\right\}$

The HJB is the limit of the Bellman equation as $\Delta \to 0$: next, the walkthrough

The Bellman equation with period length Δ

- Control: consumption c, a flow, not k'. As $\Delta \to 0$, $k' = k + \dot{k}\Delta \to k$, so what is chosen is the *rate* $\dot{k} = f(k) - \delta k - c$, i.e., c
- Utility flow $u(c)$ for Δ units of time, and capital moves by $\dot{k}\Delta$:

$$V_\Delta(k) = \max_{0 \le c \le f(k) - \delta k + k/\Delta} \left\{ u(c)\Delta + e^{-\rho\Delta} V_\Delta\big(k + [f(k) - \delta k - c]\Delta\big) \right\}$$

- Subtract $e^{-\rho\Delta} V_\Delta(k)$ from both sides and divide by Δ:

$$\frac{1 - e^{-\rho\Delta}}{\Delta} V_\Delta(k) = \max_{0 \le c \le f(k) - \delta k + k/\Delta} \left\{ u(c) + e^{-\rho\Delta} \frac{V_\Delta\big(k + [f(k) - \delta k - c]\Delta\big) - V_\Delta(k)}{\Delta} \right\}$$

- Send $\Delta \to 0$ and *assume* $V_\Delta \to V$, differentiable: the left side $\to \rho V(k)$, and the difference quotient $\to V'(k)[f(k) - \delta k - c]$

The Hamilton-Jacobi-Bellman (HJB) equation

- The limit is the **HJB equation**:

$$\rho V(k) = \max_{c \geq 0} \Big\{ u(c) + V'(k) \big[f(k) - \delta k - c \big] \Big\}$$

- No "next period": $V'(k)\dot{k}$ is the rate at which the value changes as k moves
- The maximization is *static*: choose c given k and $V'(k)$. Often in closed form
- **Caution**: the HJB is convenient, but it is only a **necessary** condition. We assumed V exists and is differentiable, and a solution of the HJB need not be the optimum (unlike the Bellman equation, it is not a contraction). Sufficiency: optimal control

The Euler equation from the HJB (1/2)

- FOC for the control c:
$$u'(c) = V'(k)$$
the marginal utility of consuming one more unit now equals the marginal value of keeping it as capital (discrete time: $u'(c) = \beta V'(k')$)

- Envelope condition for the state k (*assume* V is twice differentiable; differentiate the HJB at the optimal c):
$$\rho V'(k) = V''(k)\dot{k} + V'(k)\big[f'(k) - \delta\big]$$

- Step 1, the "one period update" applied to the FOC: in discrete time we use the FOC at t and at $t+1$. Here the FOC holds at every instant along the optimal path, so its time derivative holds too:
$$\frac{d}{dt}u'(c(t)) = \frac{d}{dt}V'(k(t)) \quad \Longleftrightarrow \quad u''(c)\dot{c} = V''(k)\dot{k}$$

The Euler equation from the HJB (2/2)

- Step 2: substitute $V''(k)\dot{k} = u''(c)\dot{c}$ and $V'(k) = u'(c)$ into the envelope condition:

$$u''(c)\dot{c} = \big[\rho + \delta - f'(k)\big]u'(c) \quad \Longleftrightarrow \quad \frac{\dot{c}}{c} = \underbrace{\frac{u'(c)}{-cu''(c)}}_{\text{EIS=1/RRA}} \big[f'(k) - \delta - \rho\big]$$

- Consumption grows iff the net return $f'(k) - \delta$ exceeds ρ; how fast depends on the **elasticity of intertemporal substitution** (EIS), the inverse of relative risk aversion with time-separable utility. CRRA: $\frac{\dot{c}}{c} = \frac{f'(k)-\delta-\rho}{\sigma}$
- Discrete time with period length Δ and CRRA:

$$u'(c_t) = e^{-\rho\Delta}u'(c_{t+\Delta})\big[1 + (f'(k_{t+\Delta}) - \delta)\Delta\big]$$

$$\Longleftrightarrow \quad \frac{\ln c_{t+\Delta} - \ln c_t}{\Delta} = \frac{1}{\sigma}\left[\frac{\ln\big(1 + (f'(k_{t+\Delta}) - \delta)\Delta\big)}{\Delta} - \rho\right] \xrightarrow{\Delta\to 0} \frac{\dot{c}}{c} = \frac{f'(k) - \delta - \rho}{\sigma}$$

With $\Delta = 1$: $\ln\frac{c_{t+1}}{c_t} \approx \frac{1}{\sigma}\big[f'(k_{t+1}) - \delta - \rho\big]$

The dynamic system

- Two ODEs in (k, c), with CRRA utility:

$$\begin{aligned}\dot{k} &= f(k) - \delta k - c \\ \dot{c} &= \frac{c}{\sigma}\left[f'(k) - \delta - \rho\right]\end{aligned}$$

- Two boundary conditions: the initial value $k(0)$ (given) and the **TVC**

$$\lim_{t\to\infty} e^{-\rho t} u'(c(t))k(t) = 0$$

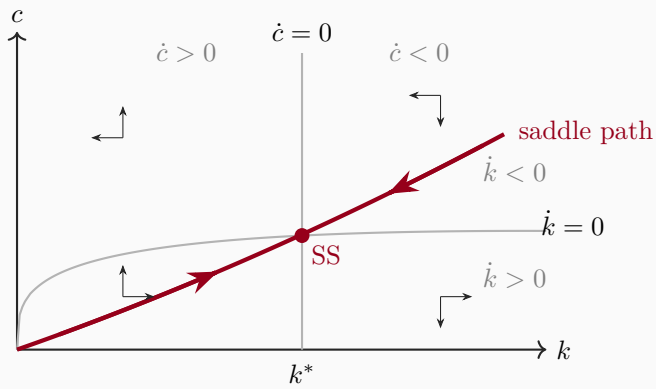
  Note: $c(0)$ is *not* given. It is chosen so the path satisfies the TVC

- Interior steady state: $\dot{k} = \dot{c} = 0$ with $c > 0$

$$f'(k^*) = \rho + \delta \quad \text{(modified golden rule)}, \qquad c^* = f(k^*) - \delta k^*$$

  ($\dot{c} = 0$ also on $c = 0$: the corner steady states $(0, 0)$ and $(\bar{k}, 0)$ with $f(\bar{k}) = \delta\bar{k}$)

The phase diagram



- $\dot{k} = 0$ locus: $c = f(k) - \delta k$
  - above: high c, so k falls
  - below: low c, so k rises
- $\dot{c} = 0$ locus: $f'(k) = \rho + \delta$
  - left: high $f'(k)$, so c rises
  - right: low $f'(k)$, so c falls
- Given $k(0)$, choose $c(0)$ on the **saddle path**: the only path that converges to the SS

Why only the saddle path?

- $c(0)$ **above** the saddle path: **infeasible**. Following the two ODEs, k hits 0 in finite time while $c > 0$, and continuing would require $k < 0$ (dropping c to 0 there would break the Euler equation)
- $c(0)$ **below** the saddle path: feasible but **not optimal**. $c \to 0$ while $k \to \bar{k} > 0$ ($f(\bar{k}) = \delta\bar{k}$): with Inada ($u'(0) = \infty$), eating a little of that capital raises utility a lot. Too much saving (overaccumulation). Formally, the TVC fails: beyond the golden rule ($f'(k) < \delta$), $u'(c)$ grows at rate $\rho + \delta - f'(k) > \rho$
- The saddle path is the policy function $c = c(k)$, the same object as the solution of the HJB. Next: optimal control shows it is the unique optimum
- Locally, the system has one stable and one unstable root linearization

Summary: the recipe

1. Write the discrete-time problem with period length Δ, then send $\Delta \to 0$
   - laws of motion become ODEs: $x' = x + g\Delta$ becomes $\dot{x} = g$
   - the Bellman equation becomes the HJB: $[e^{-\rho\Delta}V(x') - V(x)]/\Delta$ becomes $V'(x)\dot{x} - \rho V(x)$
2. FOC and envelope condition, then differentiate the FOC along the optimal path: an ODE for the control (the Euler equation)
3. Analyze the ODE system: steady state, phase diagram, saddle path
   - the phase diagram is qualitative: the actual policy $c(k)$ needs computation
   - e.g., **shooting** (guess $c(0)$, integrate the ODEs, adjust until the path converges to the SS), or solve the HJB on a grid (finite differences)

But all of this was hand-waving: is V differentiable? Does the limit of the maximizers solve the limit problem? **Optimal control** gives conditions we can check directly

Optimal Control (a crash course)

The problem

- A state $x(t)$, a control $y(t)$, a flow payoff v, and a law of motion g:

$$\max_{\{y(t)\}_{t\geq 0}} \int_0^\infty e^{-\rho t} v\big(x(t), y(t)\big) dt$$
$$\text{s.t.} \quad \dot{x}(t) = g\big(x(t), y(t)\big), \quad x(0) = x_0 \text{ given}$$

  with v, g continuously differentiable; we focus on interior solutions

- NGM: $x = k$, $y = c$, $v(k, c) = u(c)$, $g(k, c) = f(k) - \delta k - c$

- Define the (current-value) **Hamiltonian**, with a **costate** $\mu(t)$ for the state:

$$H(x, y, \mu) := v(x, y) + \mu\, g(x, y)$$

- Look familiar? It is the right side of the HJB, $v + V'(x)g$, with $V'(x)$ replaced by μ

The costate as a marginal value

- One more unit of the state x changes the Hamiltonian by

$$H_x = \underbrace{v_x}_{\text{flow payoff now}} + \underbrace{\mu\, g_x}_{\text{change in } \dot{x}\text{, valued at } \mu}$$

  so μ captures the **continuation value of a state increase**: the marginal value of the state (soon: $\mu = V'(x)$, as in the HJB)

- The **costate condition** $H_x = \rho\mu - \dot{\mu}$ is how we find μ: given the path of (x, y), a linear ODE in μ, pinned down by the TVC

- Where does the Hamiltonian come from? A Lagrangian with multiplier $e^{-\rho t}\mu(t)$ on the law of motion, integrated by parts details

Sufficient conditions

Theorem 1 (Sufficient conditions)
Suppose (x^*, y^*) *and a costate* μ^* *satisfy, for all* $t \geq 0$,

$$
\begin{aligned}
H_y(x^*, y^*, \mu^*) = v_y + \mu^* g_y &= 0 && \text{(control)} \\
H_x(x^*, y^*, \mu^*) = v_x + \mu^* g_x &= \rho\mu^* - \dot{\mu}^* && \text{(costate)} \\
\dot{x}^* &= g(x^*, y^*),\ x^*(0) = x_0 && \text{(state)} \\
\lim_{t\to\infty} e^{-\rho t}\mu^*(t)x^*(t) &= 0 && \text{(TVC)}
\end{aligned}
$$

If $H(x, y, \mu^*(t))$ *is concave in* (x, y) *for all* t, *and* $\liminf_{t\to\infty} e^{-\rho t}\mu^*(t)x(t) \geq 0$ *for any feasible path* x *(with a finite objective), then* (x^*, y^*) *is optimal; uniquely if* H *is strictly concave.*

- The $\liminf$ condition holds, e.g., if $\mu^* \geq 0$ and $x \geq 0$ proof. Control and costate conditions are also *necessary* (Pontryagin); the TVC needs care (Acemoglu, Ch. 7)

Optimal control and the HJB give the same conditions

- Compare the HJB and the Hamiltonian:

$$\rho V(x) = \max_y \left\{ v(x,y) + V'(x)\, g(x,y) \right\}$$

$$H(x,y,\mu) = v(x,y) + \mu\, g(x,y)$$

  the HJB maximizes $H(x,y,V'(x))$. Let $\mu(t) := V'(x(t))$

- FOC of the HJB: $v_y + V'(x) g_y = 0$, i.e., $H_y = 0$
- Envelope condition of the HJB, using $\dot{\mu} = V''(x)\dot{x} = V''(x) g$:

$$\rho V'(x) = v_x + V'(x) g_x + V''(x)\, g = H_x + \dot{\mu}$$

  which is the costate condition $H_x = \rho\mu - \dot{\mu}$

HJB or Hamiltonian?

- Same ODEs either way. What the optimal control theorem adds: we never need to know that V exists or is differentiable. Find (x^*, y^*, μ^*), check concavity and the TVC, done
- Where V is differentiable, $\mu(t) = V'(x(t))$: the costate is the marginal value of the state at t
- Why does research mostly use the HJB?
  - **convenient**: one equation in the state, a static maximization, no costate paths to track
  - **uncertainty**: a Poisson jump adds one term to the HJB (next section); the Hamiltonian approach gets messy
  - **interpretable**: flow value (ρV) = flow payoff + change in value; and the same grid methods solve it numerically
- But remember: the HJB alone is only necessary. Concavity and the TVC are what make a solution optimal

Example 1: the NGM

- Hamiltonian: $H(k, c, \mu) = u(c) + \mu\big[f(k) - \delta k - c\big]$
- Conditions:

$$\begin{aligned} H_c &= u'(c) - \mu = 0 \\ H_k &= \mu\big[f'(k) - \delta\big] = \rho\mu - \dot{\mu} \\ \lim_{t\to\infty} e^{-\rho t}\mu(t)k(t) &= 0 \end{aligned}$$

- With $\mu = u'(c)$, $\dot{\mu} = u''(c)\dot{c}$: the same Euler equation and TVC as before

$$\frac{\dot{c}}{c} = \frac{f'(k) - \delta - \rho}{-cu''(c)/u'(c)}, \qquad \lim_{t\to\infty} e^{-\rho t}u'(c(t))k(t) = 0$$

- Sufficiency: u, f strictly concave and $\mu = u'(c) > 0$, so H is strictly concave in (k, c); $k \geq 0$; and on the saddle path $(k, c) \to (k^*, c^*)$, so the TVC holds. So the saddle path is the unique optimum

Example 2: investment with adjustment costs

- A firm owns capital, discounts profits at the interest rate $r > 0$, and pays a convex cost $\phi(i)$ to invest i:

$$\max_{\{i(t)\}_{t\geq 0}} \int_0^\infty e^{-rt}\big[f(k) - i - \phi(i)\big]dt$$
$$\text{s.t.} \quad \dot{k} = i - \delta k, \quad k(0) = k_0 \text{ given}$$

  f strictly increasing, strictly concave, Inada; ϕ strictly convex with $\phi(0) = \phi'(0) = 0$; disinvestment $i < 0$ allowed

- Hamiltonian, calling the costate q:

$$H(k, i, q) = f(k) - i - \phi(i) + q\,(i - \delta k)$$

  strictly concave in (k, i): f and $-\phi$ are strictly concave, and the rest is linear

- $q(t) = V'(k(t))$: the value of one more unit of *installed* capital. The cost to buy a new unit of capital is 1

Tobin's q

- Control: $H_i = -1 - \phi'(i) + q = 0$, so

$$1 + \phi'(i) = q \qquad \Longrightarrow \qquad i = \iota(q) := (\phi')^{-1}(q-1), \quad \text{increasing in } q$$

  invest iff $q > 1$: **(marginal) Tobin's** q, the value of a unit of installed capital relative to its replacement cost (equals average q under the conditions in Hayashi, 1982)

- Costate: $H_k = f'(k) - \delta q = rq - \dot{q}$, i.e.,

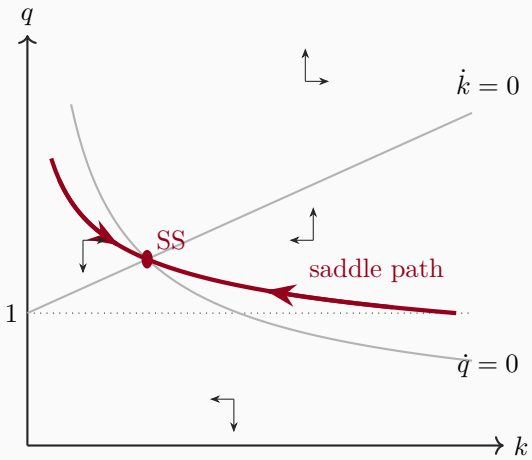
$$\dot{q} = (r+\delta)q - f'(k) \qquad \Longleftrightarrow \qquad q(t) = \int_t^\infty e^{-(r+\delta)(\tau - t)} f'(k(\tau))\, d\tau$$

  q is the present value of the future marginal products of a unit of capital, which depreciates at rate δ (if $e^{-(r+\delta)T} q(T) \to 0$, e.g., on a path to the SS)

- In the limit without adjustment costs ($\phi \to 0$): $q \to 1$ and $f'(k) = r + \delta$ at every instant: k jumps to its target at once

- Discrete time (deck 4): $q = \beta V'(k')/u'(c)$, in units of consumption

The q phase diagram



- $\dot{k} = 0$: $\iota(q) = \delta k$, upward sloping. Above: k rises
- $\dot{q} = 0$: $(r + \delta)q = f'(k)$, downward sloping. Above: q rises
- SS: $f'(k^*) = (r + \delta)\big(1 + \phi'(\delta k^*)\big)$
- k is a stock, predetermined at t; q is forward-looking, free to jump onto the saddle path. Low k: high q, high investment, and both converge

Uncertainty: Poisson Jumps

Poisson processes

- Events (a job offer, a TFP switch) arrive at a **Poisson rate** $\eta > 0$: over a short interval of length Δ,

$$\Pr(\text{one arrival}) = \eta\Delta + o(\Delta), \qquad \Pr(\text{more than one}) = o(\Delta)$$

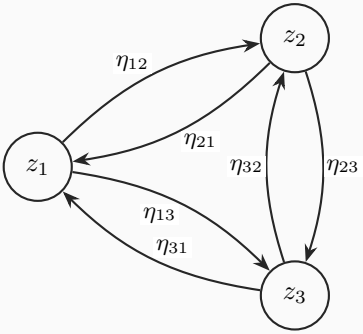
  where $o(\Delta)$ collects terms that vanish faster than Δ: $o(\Delta)/\Delta \to 0$ as $\Delta \to 0$ (e.g., Δ^2)

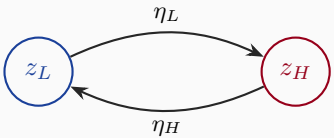
- The waiting time T until the next arrival is exponential: $\Pr(T > t) = e^{-\eta t}$, with mean $1/\eta$. Memoryless: the past waiting time tells nothing about the future

- The discrete-time counterpart: in each period of length Δ, the event happens with probability $\eta\Delta$, independently across periods

- Why Poisson? The HJB gets one extra term and needs no stochastic calculus (Brownian motion needs Itô's lemma: next semester, maybe?)

Continuous-time Markov chains





- Switching among several states works the same way: from state z_i, a switch to $z_j \neq z_i$ arrives at rate η_{ij}, a **continuous-time Markov chain**
- Discrete-time counterpart (deck 5): transition probabilities $\eta_{ij}\Delta$ per period of length Δ, and $1 - \sum_{j \neq i} \eta_{ij}\Delta$ of staying
- With two states, write η_i for the rate of leaving z_i: e.g., TFP switches between z_L and z_H

The HJB with a Poisson jump

- An exogenous state $z \in \{z_1, z_2\}$ switches from z_i to z_j, $j \neq i$, at rate η_i. With period length Δ and $x' = x + g(x, y, z_i)\Delta$:

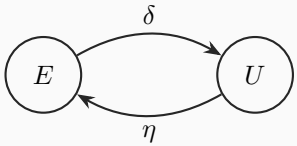
$$V(x, z_i) = \max_y \left\{ v(x, y, z_i)\Delta + e^{-\rho\Delta}\left[(1 - \eta_i\Delta)V(x', z_i) + \eta_i\Delta\, V(x', z_j)\right] \right\}$$

- Subtract $e^{-\rho\Delta}V(x, z_i)$, divide by Δ, and send $\Delta \to 0$:

$$\rho V(x, z_i) = \max_y \left\{ v(x, y, z_i) + V_x(x, z_i)\, g(x, y, z_i) + \eta_i\left[V(x, z_j) - V(x, z_i)\right] \right\}$$

- Compared with the deterministic HJB, one new term: the **expected change in value from a jump**, which happens at rate η_i and changes the value by $V(x, z_j) - V(x, z_i)$

Warm-up: employment and unemployment



employed E: income w
unemployed U: income $b < w$

- A worker consumes all income; jobs end at rate δ, and the unemployed find one at rate η. No choices at all
- Values W (employed) and U (unemployed), by the same $\Delta \to 0$ trick, with no max:

$$\rho W = w + \delta (U - W)$$
$$\rho U = b + \eta (W - U)$$

- Flow value = flow payoff + rate × jump in value. A linear system:

$$W - U = \frac{w - b}{\rho + \delta + \eta}, \qquad \rho U = b + \frac{\eta (w - b)}{\rho + \delta + \eta}$$

- As $\rho \to 0$: $\rho W, \rho U \to \frac{\eta w + \delta b}{\delta + \eta}$, average income ($\frac{\eta}{\delta + \eta}$: long-run share of time employed)

Example 1: McCall search in continuous time

- An unemployed worker receives flow income b. Job offers arrive at rate η, with wage w drawn iid from a cdf Φ. Accept and work at w forever, or keep searching
- Employed at w: $\rho W(w) = w$, so $W(w) = w/\rho$
- Unemployed: no state changes except the offer arrival, which the worker may decline:

$$\rho U = b + \eta\, \mathbb{E}\big[\max\{W(w) - U,\ 0\}\big]$$

- Accept iff $w \geq w_R := \rho U$, the **reservation wage**. Substituting,

$$w_R = b + \frac{\eta}{\rho} \int_{w_R}^{\infty} (w - w_R)\, d\Phi(w)$$

- One equation, one unknown: with $\mathbb{E}[w] < \infty$, the left side strictly increases and the right side weakly decreases in w_R, so a unique solution. w_R rises with b and with the offer rate η, and falls with ρ: more patient workers are pickier (compare PS5)

Example 2: innovation (simplified)

- A firm with quality z earns flow profit πz. R&D at intensity x costs $c(x) = \frac{1}{2}x^2$ and brings an innovation at Poisson rate x: $z \to z + \gamma$
- At rate δ, an entrant replaces the firm (creative destruction): its value drops to 0
- HJB, discounting at the interest rate r:

$$r\, v(z) = \max_{x \geq 0} \Big\{ \pi z - c(x) + \underbrace{x\big[v(z+\gamma) - v(z)\big]}_{\text{innovation, rate chosen}} + \underbrace{\delta\big[0 - v(z)\big]}_{\text{destruction}} \Big\}$$

- New here: the firm *chooses* the arrival rate of a jump. FOC:

$$c'(x) = v(z+\gamma) - v(z)$$

the marginal cost of R&D equals the value gained from innovating

Example 2: solving it

- Guess $v(z) = az + B$. Then $v(z+\gamma) - v(z) = a\gamma$, so the FOC gives a constant $x^* = a\gamma$
- Plug into the HJB and match coefficients:

$$z: \quad (r+\delta)\, a = \pi \qquad \Longrightarrow \qquad a = \frac{\pi}{r+\delta}$$

$$1: \quad (r+\delta)\, B = x^* a\gamma - c(x^*) = \tfrac{1}{2}(x^*)^2 \qquad \Longrightarrow \qquad B = \frac{(x^*)^2}{2(r+\delta)}$$

- Innovation rate:

$$x^* = \frac{\pi\gamma}{r+\delta}$$

higher with larger profits π and bigger steps γ; lower with a higher interest rate r or faster creative destruction δ, which shorten how long the firm enjoys the gain

Example 3: stochastic growth with two TFP states

- TFP $z \in \{z_L, z_H\}$ switches from z_i to z_j, $j \neq i$, at rate η_i. The planner's HJB, for $i = L, H$:

$$\rho V(k, z_i) = \max_c \Big\{ u(c) + V_k(k, z_i)\big[z_i f(k) - \delta k - c\big] + \eta_i \big[V(k, z_j) - V(k, z_i)\big] \Big\}$$

- FOC: $u'(c_i(k)) = V_k(k, z_i)$, a consumption policy for each TFP state
- Envelope condition, and $\frac{d}{dt} V_k(k, z_i) = V_{kk}(k, z_i)\dot{k}$ between jumps:

$$\frac{d}{dt} u'(c_i) + \eta_i \big[u'(c_j) - u'(c_i)\big] = \big[\rho + \delta - z_i f'(k)\big] u'(c_i)$$

- The left side is the **expected** rate of change of marginal utility: the drift plus the expected jump. A stochastic Euler equation, as in deck 5

Example 3: what the jump term does

- With CRRA, between jumps in state i:

$$\frac{\dot{c}_i}{c_i} = \frac{1}{\sigma}\left[z_i f'(k) - \delta - \rho + \eta_i\left(\frac{u'(c_j(k))}{u'(c_i(k))} - 1\right)\right]$$

- In state H, a switch to L makes c jump down (assuming $c_L(k) < c_H(k)$), so $u'(c_L) > u'(c_H)$: between jumps, c drifts up faster than without risk, to offset the expected fall at a switch (even with quadratic u)

- The expected growth of c, *including* jumps, exceeds the riskless rate only because $u''' > 0$: **precautionary saving** (deck 6)

- No closed form: $c_L(k)$ and $c_H(k)$ solve a pair of ODEs in k. In practice, solve the HJB numerically on a grid (finite differences) (Achdou et al., 2022)

- A natural next step: households with Poisson income risk and a borrowing limit, an HJB for each household plus an equation for the wealth distribution

Summary

- Continuous time = discrete time with $\Delta \to 0$
  - laws of motion become ODEs; growth rates add up
  - the Bellman equation becomes the HJB: convenient, but only a necessary condition
- Optimal control: the Hamiltonian is the HJB's right side with V' replaced by the costate μ; concavity and the TVC make the conditions sufficient
- The costate is the marginal value of the state: in the NGM $\mu = u'(c)$, in investment problems $\mu = q$, Tobin's q
- Poisson risk adds one term to the HJB: rate $\times$ jump in value

Readings

- Acemoglu Ch.7 (optimal control), Ch.8 (NGM in continuous time)
- AKMM Ch.9
- Krueger Ch.9
- Achdou, Han, Lasry, Lions, and Moll (2022), "Income and Wealth Distribution in Macroeconomics: A Continuous-Time Approach," *Review of Economic Studies*

Appendix

Local dynamics around the steady state

go back

- Linearize the NGM system (CRRA) at (k^*, c^*), where $f'(k^*) - \delta = \rho$:

$$\begin{pmatrix} \dot{k} \\ \dot{c} \end{pmatrix} \approx \underbrace{\begin{pmatrix} \rho & -1 \\ \frac{c^*}{\sigma} f''(k^*) & 0 \end{pmatrix}}_{=:J} \begin{pmatrix} k - k^* \\ c - c^* \end{pmatrix}$$

- $\det J = \frac{c^*}{\sigma} f''(k^*) < 0$: the two eigenvalues have opposite signs, one stable and one unstable. A **saddle**

- Exactly one path converges: the one along the eigenvector of the negative eigenvalue (locally, the saddle path)

- The speed of convergence is $|\nu|$, where the negative eigenvalue is $\nu = \frac{1}{2}\left[\rho - \sqrt{\rho^2 - 4\frac{c^*}{\sigma} f''(k^*)}\right] < 0$

Where the Hamiltonian comes from

go back

- Put a multiplier $\lambda(t) := e^{-\rho t}\mu(t)$ on the law of motion and integrate $\int \lambda \dot{x}\, dt$ by parts:

$$\begin{aligned}\mathcal{L} &= \int_0^\infty \Big[e^{-\rho t} v(x,y) + \lambda\big(g(x,y) - \dot{x}\big)\Big] dt \\ &= \int_0^\infty \Big[\underbrace{e^{-\rho t} v(x,y) + \lambda\, g(x,y)}_{=e^{-\rho t} H(x,y,\mu)} + \dot{\lambda} x\Big] dt + \lambda(0)x(0) - \lim_{t\to\infty} \lambda(t)x(t)\end{aligned}$$

- Pointwise FOCs for $y(t)$ and $x(t)$: $H_y = 0$ and $e^{-\rho t} H_x + \dot{\lambda} = 0$
- Since $\dot{\lambda} = e^{-\rho t}(\dot{\mu} - \rho\mu)$, the second is $H_x = \rho\mu - \dot{\mu}$; the boundary term gives the TVC $\lim_{t\to\infty} e^{-\rho t}\mu(t)x(t) = 0$
- Same logic as the discrete-time Lagrangian with $\Delta \to 0$ (deck 2)

Proof of the sufficient conditions

go back

Proof.

Let $\lambda^* := e^{-\rho t}\mu^*$, so $\dot{\lambda}^* = e^{-\rho t}(\dot{\mu}^* - \rho\mu^*) = -e^{-\rho t}H_x^*$. For any feasible (x, y), concavity of H and $H_y^* = 0$ give

$$v + \mu^* g \le v^* + \mu^* g^* + H_x^*(x - x^*) \quad \Longrightarrow \quad e^{-\rho t}(v - v^*) \le \lambda^*(g^* - g) + \dot{\lambda}^*(x^* - x)$$

Integrate, use $g = \dot{x}$, and integrate the last term by parts:

$$\begin{aligned}\int_0^\infty e^{-\rho t}(v - v^*)dt &\le \int_0^\infty \lambda^*(\dot{x}^* - \dot{x})dt + \Big[\lambda^*(x^* - x)\Big]_0^\infty - \int_0^\infty \lambda^*(\dot{x}^* - \dot{x})dt \\ &= \lim_{t\to\infty} e^{-\rho t}\mu^*(t)x^*(t) - \liminf_{t\to\infty} e^{-\rho t}\mu^*(t)x(t) \le 0,\end{aligned}$$

using $x(0) = x^*(0)$, the TVC, and the last condition of the theorem. □

Calculus of variations (FYI)

- The "old-school" method: choose the path $x(t)$ itself,

$$\max_{x(t)} \int_0^T F\big(t, x(t), \dot{x}(t)\big)dt, \quad x(0) = x_0$$

  NGM: $F(t, k, \dot{k}) = e^{-\rho t} u\big(f(k) - \delta k - \dot{k}\big)$

- Perturb the optimal path by $ah(t)$ with $h(0) = 0$; at $a = 0$ the derivative must vanish. Integrating by parts gives the **Euler-Lagrange equation** and a TVC:

$$F_x = \frac{d}{dt} F_{\dot{x}}, \qquad F_{\dot{x}}\big(T, x(T), \dot{x}(T)\big) = 0 \quad \text{(if } x(T) \text{ is free)}$$

- Sufficient if F is concave in $(x, \dot{x})$

- A special case of optimal control with $\dot{x} = y$: with the present-value Hamiltonian $F + \lambda y$, the conditions $F_y + \lambda = 0$ and $\dot{\lambda} = -F_x$ give $F_x = \frac{d}{dt} F_{\dot{x}}$

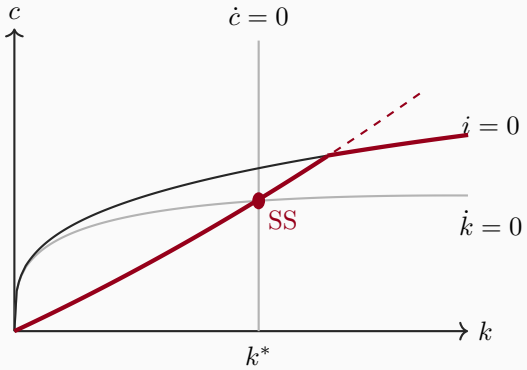
Constraints: irreversible investment (FYI)

- Planner with $c + i = f(k)$, $\dot{k} = i - \delta k$, and **irreversible investment** $i \geq 0$ (deck 4 in continuous time)
- Augmented Hamiltonian with a multiplier $\omega \geq 0$ on $i \geq 0$:

$$H(k, i, \mu, \omega) = u\big(f(k) - i\big) + \mu(i - \delta k) + \omega i$$

  strictly concave in (k, i), since $u(f(k) - i)$ is
- Conditions: $u'(c) = \mu + \omega$, $\omega \geq 0$, $\omega i = 0$; $H_k = u'(c)f'(k) - \delta\mu = \rho\mu - \dot{\mu}$; TVC
- When $i > 0$: $\omega = 0$, $\mu = u'(c)$, the usual Euler equation. When $i = 0$: $c = f(k)$ and $\dot{k} = -\delta k$
- $\mu > 0$: if $\mu \leq 0$, then $\omega > 0$, so $i = 0$ and $k = k_0 e^{-\delta t}$ from then on, while $\dot{\mu} < 0$ keeps μ falling at rate above $\rho + \delta$; so $e^{-\rho t}\mu k$ stays negative and decreasing, violating the (necessary) TVC

Irreversible investment: the phase diagram (FYI)



- $\dot{k} = 0$ and $\dot{c} = 0$ loci unchanged
- New curve $i = 0$, i.e., $c = f(k)$. Above it: $i < 0$, infeasible
- Optimal path: the saddle path where it lies below $c = f(k)$; at higher k, $i = 0$ ($c = f(k)$) and capital depreciates until it reaches the saddle path