ECON 720: Advanced Macroeconomic Theory I

Extensions and Variants of Stochastic Neoclassical Growth Model

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Overview

Common core: households maximize expected lifetime utility given prices, and their FOCs set (probability-weighted, discounted) MRS equal to prices. What changes: which markets exist, and who trades with whom

1. Risk sharing in a pure-exchange economy with heterogeneous agents
   With complete markets, who bears the risk? Only aggregate risk remains
2. Uninsurable risk: self-insurance and precautionary saving
   When risk cannot be shared, households insure themselves by saving
3. Asset pricing with complete markets
   The same MRS that shares risk also prices every asset
4. The equity premium puzzle (optional)
   Do these prices fit the data?
5. Real business cycles (RBC) (next semester)
   The stochastic NGM as a model of business cycles

Risk-Sharing with Pure Exchange

Setup

We step away from stochastic NGM and work with **pure-exchange** (no production) economies for now.

- $t = 0, 1, 2, \ldots$
- A finite set of events S
- Event history $s^t \in S^t$, unconditional probability $\pi_t(s^t) > 0$
- No uncertainty at $t = 0$: known and fixed $s_0 \in S$, so $\pi_0(s_0) = 1$
- Agents: $N \geq 1$ households or agents, each with expected lifetime utility

$$U(c^i) = U(\{\{c_t^i(s^t)\}\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi_t(s^t) u(c_t^i(s^t))$$

where $\beta \in (0, 1)$, $u' > 0$, $u'' < 0$, Inada conditions: $u'(0) = \infty$, $u'(\infty) = 0$

Setup

- $\forall t, s^t$, agent i receives an endowment (or income) $e_t^i(s^t) \geq 0$ of period-t, history-s^t consumption good
- Let the total resource/endowment at t, s^t be

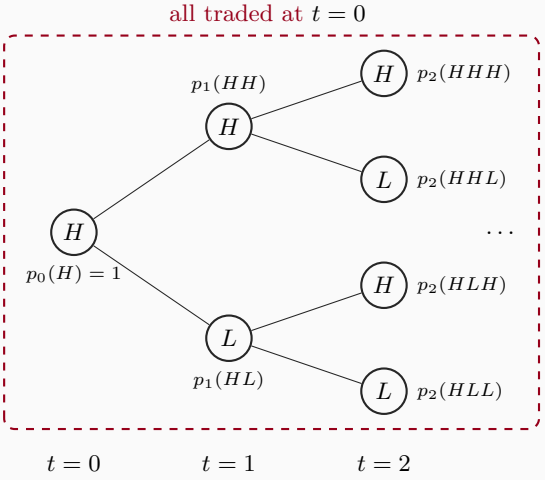
$$e_t(s^t) = \sum_{i=1}^{N} e_t^i(s^t)$$

  - if $e_t(s^t)$ remains constant across s^t: **no aggregate (income) risk**
  - otherwise: **aggregate risk**

- Example: 2 agents, 3 events, no aggregate risk:

$$(e_t^1(s^t), e_t^2(s^t)) = \begin{cases} (1,0) & \text{if } s_t = 1 \\ (0,1) & \text{if } s_t = 2 \\ (0.5, 0.5) & \text{if } s_t = 3 \end{cases}$$

Arrow-Debreu markets: one price per history



- Each node s^t: a market for the history-s^t good, at price $p_t(s^t)$ in units of the time-0 good
- All markets open at $t = 0$, before any uncertainty is resolved: one lifetime budget constraint per agent
- After $t = 0$: no more trading, only deliveries

Arrow-Debreu market structure

All markets open at $t = 0$ *before* any risk is resolved

Definition 1 (ADE)
An **Arrow-Debreu equilibrium** consists of (Arrow-Debreu) prices $\{\{p_t(s^t)\}_{s^t \in S^t}\}_{t=0}^{\infty}$ and allocations $\{\{(c_t^i(s^t))_{i=1,\ldots,N}\}_{s^t \in S^t}\}_{t=0}^{\infty}$, such that

- (Optimization) given prices, $\{\{c_t^i(s^t)\}_{s^t \in S^t}\}_{t=0}^{\infty}$ solves agent i's problem, $\forall i$:

$$\max_{\{\{c_t^i(s^t)\}_{s^t \in S^t}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi_t(s^t) u(c_t^i(s^t))$$

$$\text{subject to } \sum_{t=0}^{\infty} \sum_{s^t \in S^t} p_t(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t \in S^t} p_t(s^t) e_t^i(s^t),$$

$$c_t^i(s^t) \geq 0, \ \forall t, s^t.$$

- (Market clearing) all markets opening at $t = 0$ clear: $\sum_{i=1}^{N} c_t^i(s^t) = e_t(s^t), \ \forall t, s^t$.

Remarks

- Each agent i has only 1 budget constraint
- No probabilities in the budget constraint!
- Can normalize only 1 price to be 1, e.g., $p_0 = p_0(s_0) = 1$
- Resource constraints/market clearing conditions hold in every period and history

Equilibrium characterization

FOCs for agent i:

$$\beta^t \pi_t(s^t) u'(c_t^i(s^t)) = \lambda^i p_t(s^t), \qquad \forall t, s^t,$$

which implies the following

- with $p_0 = 1$ and $\pi_0(s_0) = 1$, agent i's multiplier on budget constraint satisfies (dropping s_0 in $c_0^i(s_0)$)

  $$\lambda^i = u'(c_0^i), \quad \forall i$$

  interpretation: marginal value of increasing agent i's *lifetime wealth* (evaluated at $t = 0$) is agent i's marginal utility at $t = 0$

- **Arrow-Debreu price** at (t, s^t) satisfies

  $$p_t(s^t) = \beta^t \pi_t(s^t) \frac{u'(c_t^i(s^t))}{u'(c_0^i)}, \quad \forall i,$$

  equivalently...

Equilibrium characterization (cont.)

- given common β, common beliefs $\pi_t(s^t)$, and the common market price $p_t(s^t)$:

$$u'(c_t^i(s^t)) = \underbrace{u'(c_0^i)}_{\lambda^i,\text{ depends on } i \text{ but not } t,s^t} \times \underbrace{\frac{p_t(s^t)}{\beta^t \pi_t(s^t)}}_{\text{depends on } t,s^t \text{ but not } i}$$

it further implies:

- $\forall s^t, s^\tau$, a common MRS b/w s^t and s^τ *across all agents*

$$MRS(s^t, s^\tau) = \frac{\beta^t \pi_t(s^t) u'(c_t^i(s^t))}{\beta^\tau \pi_\tau(s^\tau) u'(c_\tau^i(s^\tau))} = \frac{p_t(s^t)}{p_\tau(s^\tau)}, \quad \forall i$$

- $\forall i, j$, a constant cross-agent ratio of marginal utility *in all states*

$$\frac{u'(c_t^i(s^t))}{u'(c_t^j(s^t))} = \frac{u'(c_0^i)}{u'(c_0^j)}$$

- and **perfect risk sharing** (next slide)

Perfect risk sharing

Perfect risk sharing, with a strictly concave u, a common β, and common beliefs $\{\pi_t(s^t)\}$: agent i's consumption at (t, s^t) depends on aggregate endowment/income $e_t(s^t)$ but not on own $e_t^i(s^t)$

- From the previous slide:

$$c_t^i(s^t) = (u')^{-1}\left(u'(c_0^i)\frac{p_t(s^t)}{\beta^t \pi_t(s^t)}\right) \qquad \text{(strict concavity of } u\text{)}$$

$$\overset{\text{market clearing}}{\Longrightarrow} \quad e_t(s^t) = \sum_i c_t^i(s^t) = \sum_i (u')^{-1}\left(u'(c_0^i)\frac{p_t(s^t)}{\beta^t \pi_t(s^t)}\right), \quad \forall t, s^t$$

  which means the equilibrium price $p_t(s^t)$ depends on aggregate $e_t(s^t)$ (given the λ^i's)

- So $c_t^i(s^t)$ depends on 2 things only: (i) aggregate income at (t, s^t) and (ii) agent i's *lifetime wealth*, but not on own $e_t^i(s^t)$
- Only aggregate risk is borne: each agent i's consumption at t varies across s^t only if aggregate income is risky at t
- Perfect risk sharing does NOT mean equal consumption across agents!

Special case: CRRA utility

- With $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, agents share aggregate income with constant proportions s.t.

$$\left(\frac{c_t^i(s^t)}{c_t^j(s^t)}\right)^{-\sigma} = \frac{\lambda^i}{\lambda^j},\ \forall t, s^t, \quad \text{which implies} \quad \frac{c_t^i(s^t)}{c_0^i} = \frac{e_t(s^t)}{e_0} \quad \Longleftrightarrow \quad c_t^i(s^t) = \frac{c_0^i}{e_0}\, e_t(s^t)$$

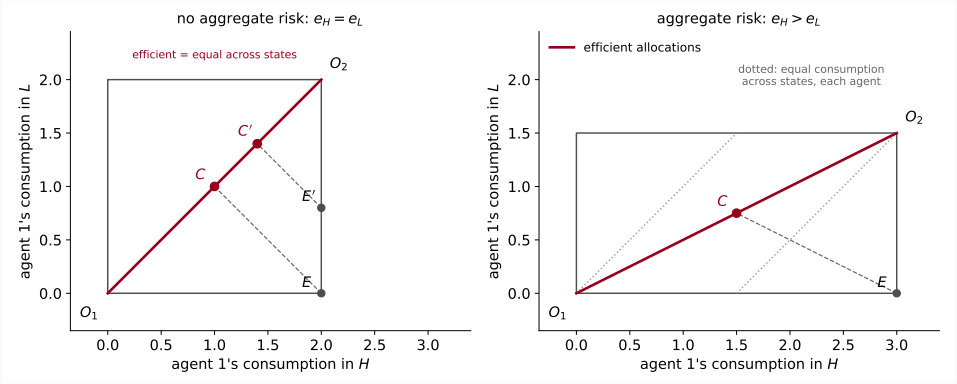
- Equilibrium prices:

$$p_t(s^t) = \beta^t \pi_t(s^t) \left(\frac{e_t(s^t)}{e_0}\right)^{-\sigma}$$

  History-s^t good is more valuable if it is (i) nearer, (ii) more likely, (iii) a time of scarce resources

- Prices depend on aggregate endowments only: a representative agent model with the *same* CRRA utility and endowment $\{e_t(s^t)\}$ gives the same prices (linear aggregation, as in deck 4's FYI slide)

Perfect risk sharing in an Edgeworth box



- Recall the Edgeworth box from micro: the red line is the contract curve. One period, two states H, L, log utility, $\pi_H = \pi_L = \frac{1}{2}$. Constant shares make it the box's diagonal
- Perfect risk sharing $\neq$ equal consumption: where you land (C vs C') depends on your endowment, i.e., your wealth

Pareto efficiency

Is the ADE outcome Pareto efficient? YES, by the first welfare theorem.

Definition 2 (Feasibility)
An allocation $\{\{(c_t^i(s^t))_{i=1,\ldots,N}\}_{s^t \in S^t}\}_{t=0}^{\infty}$ is feasible if $c_t^i(s^t) \geq 0$, $\forall i, t, s^t$, and $\sum_{i=1}^{N} c_t^i(s^t) = e_t(s^t),\ \forall t, s^t$.

Definition 3 (Pareto efficiency)
An allocation $(c^i)_{i=1,\ldots,N} = \{\{(c_t^i(s^t))_{i=1,\ldots,N}\}_{s^t \in S^t}\}_{t=0}^{\infty}$ is Pareto efficient if there does not exist another feasible allocation $(\widetilde{c^i})_{i=1,\ldots,N} = \{\{(\widetilde{c^i}_t(s^t))_{i=1,\ldots,N}\}_{s^t \in S^t}\}_{t=0}^{\infty}$ such that

$$
\begin{aligned}
U(\widetilde{c^i}) &\geq U(c^i) \quad \text{for all } i, \\
U(\widetilde{c^i}) &> U(c^i) \quad \text{for at least one } i.
\end{aligned}
$$

Sequential markets

- Suppose markets open sequentially in each t, s^t
- **Without any financial assets**, no trade ever occurs (**autarky**), agent consumes own endowment only

$$c_t^i(s^t) = e_t^i(s^t), \ \forall t, s^t$$

  in general, not equivalent to ADE (unless agents are identical)

- **Q:** When is SME outcome equivalent to ADE outcome?
- **A:** If there are "enough" financial assets to transfer income (but never resources) across time and event history
- What is a **financial asset**? The *holder*'s claim over the *issuer*'s future income at certain times in certain events
  - Simplest asset: one-period (state-contingent) IOUs, aka **Arrow securities**

Sequential markets

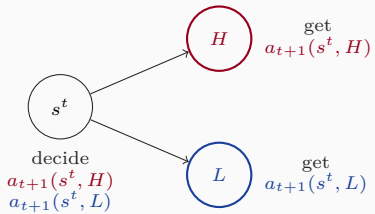
- Minimal requirement: just enough assets to cover *every possible event next period*
- $q_t(s^t, s_{t+1})$: price at t, s^t of an asset/contract that pays 1 unit of consumption good IFF s_{t+1} realizes at $t+1$, $s_{t+1} \in S$
- $a^i_{t+1}(s^t, s_{t+1})$: quantity of these assets (**Arrow securities**) held by agent i
  - > 0: buyer/holder/lender, pays at s^t and gets paid at s_{t+1}
  - < 0: seller/issuer/borrower, gets paid at s^t and pays at s_{t+1}
- Agent i's budget constraint at t, s^t:

$$c^i_t(s^t) + \sum_{s_{t+1} \in S} q_t(s^t, s_{t+1}) a^i_{t+1}(s^t, s_{t+1}) \leq e^i_t(s^t) + a^i_t(s^t)$$

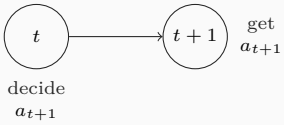
- Without loss, assume $a^i_0(s_0) = 0$, $\forall i, s_0$

What Arrow securities are and do

with uncertainty



without uncertainty



- Arrow securities: the smallest building blocks of any financial asset (think: the smallest Lego pieces)
- Example: income 2 in H, 0 in L. How to transfer x units of income from H to L?
  - buy at s^t: $a_{t+1}(s^t, L) = x$ (moves x to L)
  - sell at s^t: $a_{t+1}(s^t, H) = -x$ (moves x from H)
  - caution: prices may differ so $c_t(s^t)$ may change
- One security per successor state: any payoff pattern $(a_{t+1}(s^t, H), a_{t+1}(s^t, L))$ is possible. Markets are **complete**
- A risk-free bond pays the same in H and L: it moves goods over time, not across states. Without uncertainty, that is all we need: saving/lending (deck 4)

SME with a full set of Arrow securities

Definition 4 (SME)
A **sequential markets equilibrium** consists of asset prices $\{\{q_t(s^t, s_{t+1})\}_{s^t, s_{t+1}}\}_t$ and allocations $\{\{(c_t^i(s^t), a_{t+1}^i(s^t, s_{t+1}))_i\}_{s^t, s_{t+1}}\}_t$ such that

- (optimization) given prices, $\{\{(c_t^i(s^t), a_{t+1}^i(s^t, s_{t+1}))_i\}_{s^t, s_{t+1}}\}_t$ solves i's problem, $\forall i$

$$\max_{\{\{c_t^i(s^t), a_{t+1}^i(s^t,\cdot)\}_{s^t \in S^t}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi_t(s^t) u(c_t^i(s^t)), \quad \text{subject to}$$

$$c_t^i(s^t) + \sum\nolimits_{s_{t+1} \in S} q_t(s^t, s_{t+1}) a_{t+1}^i(s^t, s_{t+1}) \leq e_t^i(s^t) + a_t^i(s^t), \ \forall t, s^t$$

$$c_t^i(s^t) \geq 0, \ a_{t+1}^i(s^t, s_{t+1}) \geq -\overline{A}_{t+1}^i(s^t, s_{t+1}), \ \forall t, s^t$$

- (market clearing) all markets clear at each t, s^t

$$\sum\nolimits_{i=1}^{N} c_t^i(s^t) = e_t(s^t), \quad \sum\nolimits_{i=1}^{N} a_t^i(s^t) = 0$$

Remarks

- With a full set of Arrow securities, agents can transfer income freely across time and state at prices $q_t(s^t, s_{t+1})$
- "Natural debt limit" $\overline{A}^i_{t+1}(s^t, s_{t+1})$ form rules out the Ponzi scheme but never binds
- Note the condition for asset market clearing:

$$\sum_{i=1}^{N} a_t^i(s^t) = 0, \ \forall t, s^t$$

no net supply of any financial asset: if some agents choose to hold assets, some other agents must issue these assets.

  - e.g., $N = 2$, if agent 1 buys $a_t^1(s^{t-1}, s_t) > 0$ at s^{t-1}, agent 2 sells/issues $a_t^2(s^{t-1}, s_t) = -a_t^1(s^{t-1}, s_t) < 0$ at s^{t-1}

Equilibrium characterization

- FOCs for agent i, $\forall t, s^t$:

$$\begin{aligned}\beta^t \pi_t(s^t) u'(c_t^i(s^t)) =& \lambda_t^i(s^t)\\ \lambda_t^i(s^t) q_t(s^t, s_{t+1}) =& \lambda_{t+1}^i(s^t, s_{t+1})\end{aligned}$$

- Combining them yields the Euler equation:

$$\begin{aligned}\pi_t(s^t) u'(c_t^i(s^t)) q_t(s^t, s_{t+1}) =& \beta \pi_{t+1}(s^t, s_{t+1}) u'(c_{t+1}^i(s^t, s_{t+1}))\\ \iff \quad q_t(s^t, s_{t+1}) =& \beta \underbrace{\frac{\pi_{t+1}(s^t, s_{t+1})}{\pi_t(s^t)}}_{=\pi(s_{t+1}|s^t)} \frac{u'(c_{t+1}^i(s^t, s_{t+1}))}{u'(c_t^i(s^t))}\end{aligned}$$

- It also implies

$$\frac{u'(c_{t+1}^i(s^t, s_{t+1}))}{u'(c_t^i(s^t))} = \frac{u'(c_{t+1}^j(s^t, s_{t+1}))}{u'(c_t^j(s^t))}, \ \forall t, s^t, s_{t+1}$$

ADE = SME with a full set of Arrow securities

- Can check that ADE allocations and SME consumptions are the same
- **Markets are complete** and risk sharing is perfect
- Equilibrium prices satisfy

$$\begin{aligned} q_t(s^t, s_{t+1}) =& \frac{p_{t+1}(s^t, s_{t+1})}{p_t(s^t)} \\ p_{t+1}(s^{t+1}) =& \underbrace{p_0(s_0)}_{=1} \times q_0(s_0, s_1) \times \ldots \times q_t(s^t, s_{t+1}) \end{aligned}$$

- $q_t(s^t, s_{t+1})$: value of 1 unit of good at s^{t+1} in units of history-s^t good
- $p_{t+1}(s^{t+1})$: value of 1 unit of good at s^{t+1} in units of numeraire at $t = 0$

Recursive formulation with Arrow securities

- Assume s_t is Markov with $\pi(s' \mid s)$, and endowments depend on the current event only: $e_t^i(s^t) = e^i(s_t)$
- Agent i's states: own assets a, the event s, and the wealth distribution $A = (A^1, \ldots, A^N)$ ("big K, little k", deck 4), since prices may depend on it: $q(s' \mid s, A)$
- Agent i perceives the law of motion $A'(s') = G(s, A, s')$ and solves

$$V^i(a, s, A) = \max_{c, \{a'(s')\}_{s' \in S}} \left\{ u(c) + \beta \sum_{s' \in S} \pi(s' \mid s) V^i\big(a'(s'), s', G(s, A, s')\big) \right\}$$

$$\text{s.t. } c + \sum_{s' \in S} q(s' \mid s, A)\, a'(s') \le e^i(s) + a, \qquad a'(s') \ge -\overline{A}^i\big(s', G(s, A, s')\big)$$ form

with policies $c = g^{c,i}(a, s, A)$ and $a'(s') = g^{a,i}(a, s, A, s')$

Recursive competitive equilibrium with Arrow securities

Definition 5 (RCE)
A **recursive competitive equilibrium** consists of a pricing kernel $q(s' \mid s, A)$, a law of motion G, value functions $\{V^i\}_i$, and policies $\{g^{c,i}, g^{a,i}\}_i$ such that

- (optimization) given q and G, V^i solves i's Bellman equation with policies $g^{c,i}, g^{a,i}$;
- (consistency) $G^i(s, A, s') = g^{a,i}(A^i, s, A, s')$, $\forall i, s, A, s'$;
- (market clearing) $\forall s$ and A with $\sum_i A^i = 0$,

$$\sum_{i=1}^{N} g^{c,i}(A^i, s, A) = \sum_{i=1}^{N} e^i(s), \qquad \sum_{i=1}^{N} g^{a,i}(A^i, s, A, s') = 0, \ \forall s' \in S$$

- With complete markets, along the path from $A_0 = 0$, A depends on s only and $q(s' \mid s) = \beta\pi(s' \mid s)\, u'(c^i(s'))/u'(c^i(s))$ with $c^i(s) = \phi^i(e(s))$: no need to track A (Ljungqvist and Sargent, Ch. 8)
- Caution: with incomplete markets, no such shortcut: A is a genuine state

A simple example

- Two agents, $i = 1, 2$, with $u(c) = \ln c$
- No uncertainty at $t = 0$:

$$(e_0^1, e_0^2) = (2, 0)$$

  agent 1 gets rich first

- Endowments afterwards are determined by an iid flip of a coin, $\forall t \geq 1$

$$(e_t^1(s^t), e_t^2(s^t)) = \begin{cases} (2,0) & \text{if } s_t = 1 \text{ (head, w.p. } 1/2) \\ (0,2) & \text{if } s_t = 2 \text{ (tail, w.p. } 1/2) \end{cases}$$

  so $\pi_t(s^t) = (1/2)^t$, $\forall t, s^t$

A simple example: Arrow-Debreu equilibrium

- AD prices: $p_t(s^t) = \beta^t \pi_t(s^t) \left(\frac{e_t(s^t)}{e_0}\right)^{-1} = \left(\frac{\beta}{2}\right)^t, \ \forall t, s^t \ (p_0 = 1)$
- Allocations: $c_t^i(s^t) = \frac{e_t(s^t)}{e_0} c_0^i = c_0^i \ \forall t, s^t$
- So agent i's lifetime expenditure becomes

$$\sum_{t=0}^{\infty} \sum_{s^t \in \{1,2\}^t} p_t(s^t) c_t^i(s^t) = \sum_{t=0}^{\infty} 2^t \times \left(\frac{\beta}{2}\right)^t \times c_0^i = \frac{c_0^i}{1-\beta}$$

- Each agent's lifetime income then becomes

$$i=1: \quad 2 + \sum_{t=1}^{\infty} \sum_{(s^{t-1}, s_t = 1)} \left(\frac{\beta}{2}\right)^t \times 2 = 2 + \sum_{t=1}^{\infty} 2^{t-1} \times \left(\frac{\beta}{2}\right)^t \times 2 = 2 + \frac{\beta}{1-\beta} \implies c_0^1 = 2 - \beta$$

$$i=2: \quad 0 + \sum_{t=1}^{\infty} \sum_{(s^{t-1}, s_t = 2)} \left(\frac{\beta}{2}\right)^t \times 2 = \sum_{t=1}^{\infty} 2^{t-1} \times \left(\frac{\beta}{2}\right)^t \times 2 = \frac{\beta}{1-\beta} \implies c_0^2 = \beta < c_0^1$$

A simple example: sequential markets equilibrium

with a full set of Arrow securities

- SM prices:

$$q_t(s^t, s_{t+1}) = \beta \frac{\pi_{t+1}(s^t, s_{t+1})}{\pi_t(s^t)} \left(\frac{e_{t+1}(s^t, s_{t+1})}{e_t(s^t)} \right)^{-1} = \beta \frac{(1/2)^{t+1}}{(1/2)^t} = \frac{\beta}{2}, \ \forall t, s^t$$

- SM consumptions: $c_0^1 = c_t^1(s^t) = 2 - \beta$, $c_0^2 = c_t^2(s^t) = \beta$

- SM asset positions, $\forall t, s^t$:

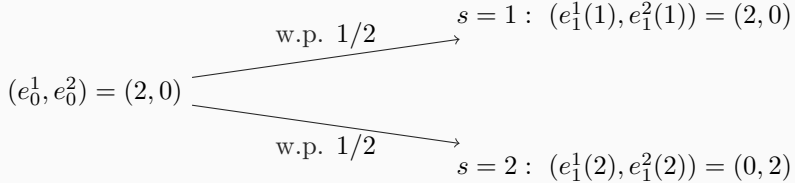
$$a_{t+1}^1(s^t, s_{t+1} = 1) = -a_{t+1}^2(s^t, s_{t+1} = 1) = 0$$
$$a_{t+1}^1(s^t, s_{t+1} = 2) = -a_{t+1}^2(s^t, s_{t+1} = 2) = 2$$

  Interpretation: regardless of current event history s^t, agent 1 always buys 2 units of Arrow security that pays off if $s_{t+1} = 2$, and agent 2 sells/issues them.

- What if the markets are **incomplete**?

An even simpler example

- Consider a 2-period version of the model with no uncertainty at $t = 0$



- **Arrow-Debreu equilibrium** outcome:

$$p_1(1) = p_1(2) = \frac{\beta}{2}$$

$$c_0^1 = c_1^1(1) = c_1^1(2) = \frac{2+\beta}{1+\beta} \qquad c_0^2 = c_1^2(1) = c_1^2(2) = \frac{\beta}{1+\beta}$$

An even simpler example

- **Sequential markets equilibrium** outcome with a full set of Arrow securities

$$q_1(1) = q_1(2) = \frac{\beta}{2}$$

$$c_0^1 = c_1^1(1) = c_1^1(2) = \frac{2+\beta}{1+\beta} \qquad c_0^2 = c_1^2(1) = c_1^2(2) = \frac{\beta}{1+\beta}$$

$$a_1^1(1) = -a_1^2(1) = -\frac{\beta}{1+\beta} \qquad a_1^1(2) = -a_1^2(2) = \frac{2+\beta}{1+\beta}$$

- **Incomplete market** with only 1 Arrow security for $s = 1$:
  - Agent 1 cannot have extra income at $s = 2$ and consumes nothing
- **Incomplete market** with only a risky-free bond (borrowing/lending):
  - Agent 2 wants to borrow at $t = 0$ but cannot afford repayment if $s = 1$

Self-Insurance and Precautionary Saving

Uninsurable risk

- Complete markets: consumption responds to aggregate risk only. Idiosyncratic income risk is shared away
- Risk stays **uninsurable** if markets are incomplete (e.g., a risk-free bond only), or if it is aggregate (no one to share it with)
- Then households **self-insure**: they save in the assets available, and run them down in bad times
- **Q**: do households save more when their income is riskier?
- **A**: Yes, if they are *prudent* or face *possibly binding borrowing constraints*
- To illustrate, we focus on *2-period decision problems*: one household, income risk, a risk-free bond (deck 4's consumption-saving problem, plus uncertainty)

Precautionary saving

- $t = 0, 1$; deterministic income $w_0 > 0$ at $t = 0$; income risk at $t = 1$ s.t. random $w_1 > 0$ with mean $\overline{w_1}$
- a risk-free bond (or saving/borrowing) with gross interest rate R:

$$\max_a \{u(w_0 - a) + \beta \mathbb{E}[u(Ra + w_1)]\}$$

  Euler equation:

$$u'(w_0 - a) = \beta R \, \mathbb{E}[u'(Ra + w_1)]$$

- **Prudence**: if u' is *str. convex* ($u''' > 0$), $\mathbb{E}[u'(Ra + w_1)] > u'(Ra + \overline{w_1})$ by Jensen, $\forall a$
- **Precautionary saving**: at the saving a chosen under certainty ($w_1 = \overline{w_1}$), the RHS of EE exceeds the LHS, so the household *saves more under income uncertainty*
- CRRA: $u'''(c) = \sigma(\sigma + 1)c^{-\sigma-2} > 0$, so CRRA households are prudent

Why does $u''' > 0$ lead to additional saving?

- $u'' < 0$ says risk hurts; $u''' > 0$ says risk hurts *less when richer*: for $\mathbb{E}\varepsilon = 0$,

$$\frac{d}{dc}\underbrace{\Big\{\mathbb{E}[u(c+\varepsilon)] - u(c)\Big\}}_{\text{utility change from risk } \varepsilon\ (<0)} = \mathbb{E}[u'(c+\varepsilon)] - u'(c) > 0 \iff u' \text{ convex}$$

  Saving raises future wealth under income risk so it hurts less

- Prudence $\neq$ risk aversion: quadratic u is risk averse ($u'' < 0$) but $u''' = 0$: risk hurts equally at every c, so no precautionary saving (certainty equivalence)
- Define $\hat{c}$ by $u'(\hat{c}) = \mathbb{E}[u'(Ra + w_1)]$: the household saves as if $c_1 = \hat{c}$ for sure
  - $Ra + \overline{w_1} - \hat{c} > 0$: the *precautionary premium* (Kimball, 1990)
  - Extreme prudence: $\hat{c} \to Ra + \min w_1$, only the worst outcome matters
- Cf. ambiguity aversion (micro): also worst case, but over *beliefs* (unknown π). Here π is known; the worst-case weight comes from the *shape of* u

Borrowing constraints

- Same 2-period model but no borrowing allowed

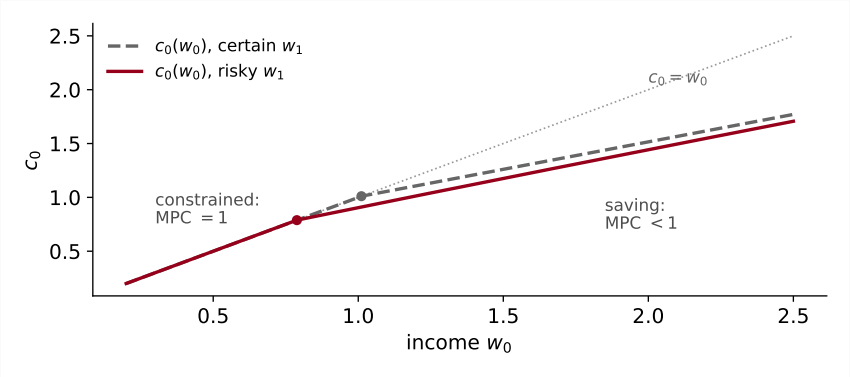
$$\max_{a \geq 0} \{u(w_0 - a) + \beta \mathbb{E}[u(Ra + w_1)]\}$$

  Euler equation:

$$u'(w_0 - a) \geq \beta R \, \mathbb{E}[u'(Ra + w_1)], \text{ with "=" if } a > 0 \quad \text{(deck 4)}$$

- Poor households (low w_0) are constrained: $a = 0$ and $c_0 = w_0$, MPC $= 1$
- Richer ones (high w_0) save: $a > 0$ and $c_0 < w_0$, MPC < 1.
- Risk $\Rightarrow$ incentives to save more, and start saving at lower w_0
- Note: with a longer horizon (finite or infinite), *a borrowing constraint that may bind in the future raises saving today* even without prudence (consumption smoothing incentives)

Borrowing constraints



CRRA $\sigma = 2$, $\beta = 0.95$, $R = 1.03$; certain $w_1 = 1$ vs. $w_1 \in \{0.6, 1.4\}$ w.p. $1/2$ each

Why it matters

- Constrained households have high MPCs: recall the large MPCs out of tax rebates (deck 4)
- Infinite horizon, $\beta R < 1$: impatience pulls assets down; precautionary saving and the fear of hitting the constraint push them up. Assets fluctuate in a bounded range: **buffer-stock saving** with a stationary distribution (deck 5) (Deaton, 1991; Carroll, 1997)
- With aggregate risk, precautionary motives are one reason why SSS $\neq$ DSS (deck 5)
- Next semester: many such households $\implies$ the wealth distribution, the interest rate, and aggregate capital are determined together in equilibrium (Aiyagari, 1994)

Asset Pricing with Complete Markets

Assembling assets from Arrow securities

- From an asset holder's POV: a financial asset X is a payment stream $\{d_t^X(s^t)\}_{t,s^t}$, in units of history-s^t goods
- **No arbitrage pricing**: with complete markets, the price of X is the value of that stream at A-D prices. After the payment at s^t,

$$\underbrace{q_t^X(s^t)}_{\text{price of } X \text{ at } t,s^t} = \sum_{\tau=1}^{\infty} \sum_{s^{t+\tau}|s^t} \underbrace{\frac{p_{t+\tau}(s^{t+\tau})}{p_t(s^t)}}_{\text{A-D prices}} d_{t+\tau}^X(s^{t+\tau})$$

- Examples
  - one-period risk-free bond: one Arrow security for each s_{t+1}, so $q_t^{RF}(s^t) = \sum_{s_{t+1}} q_t(s^t, s_{t+1})$
  - a Lucas tree (stock): dividends $d_t = e_t$, the aggregate endowment
  - a European call option on the stock at T with strike K: $d_T = \max\{q_T^S - K, 0\}$

The stochastic discount factor

- Write $q_t^X(s^t)$ recursively:

$$q_t^X(s^t) = \sum_{s_{t+1}} \underbrace{\frac{p_{t+1}(s^{t+1})}{p_t(s^t)}}_{=q_t(s^t, s_{t+1})} \left[d_{t+1}^X(s^{t+1}) + \underbrace{\sum_{\tau=1}^{\infty} \sum_{s^{t+1+\tau} | s^{t+1}} \frac{p_{t+1+\tau}(s^{t+1+\tau})}{p_{t+1}(s^{t+1})} d_{t+1+\tau}^X(s^{t+1+\tau})}_{=q_{t+1}^X(s^{t+1})} \right]$$

- Plug in $p_t(s^t) = \beta^t \pi_t(s^t) \frac{u'(c_t(s^t))}{u'(c_0(s_0))}$ (and drop event history):

$$q_t^X = \mathbb{E}_t\left[m_{t+1}(d_{t+1}^X + q_{t+1}^X)\right] \quad \Longleftrightarrow \quad 1 = \mathbb{E}_t[m_{t+1} R_{t+1}^X], \qquad \text{where } m_{t+1} := \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

  m_{t+1} is the **stochastic discount factor** (SDF), or pricing kernel

  $R_{t+1}^X := \dfrac{d_{t+1}^X + q_{t+1}^X}{q_t^X}$ is the 1-period ahead gross return. Note that $R_{t+1}^{RF} = \dfrac{1}{\mathbb{E}_t[m_{t+1}]}$

- Risk premium: $\mathbb{E}_t[R_{t+1}^X] - R_{t+1}^{RF} \approx -\text{cov}_t(m_{t+1}, R_{t+1}^X)$. Assets that pay off when consumption is scarce (m high) are worth more (deck 5: state prices)

The Equity Premium Puzzle (optional)

The equity premium puzzle

- Mehra and Prescott (1985)
- Puzzle: asset return data and asset pricing model do not agree (quantitatively)...
- "High" equity premium in the data: equity (stocks) return is on average $6 \sim 7\%$ higher than gov bond return
- Standard asset pricing model fails to generate such a high equity premium with reasonable parameter values

The equity premium puzzle

- Recall the asset-pricing formula for any asset X's return:

$$\begin{aligned} 1 =& \mathbb{E}_t[m_{t+1} R^X_{t+1}] \\ =& cov_t(m_{t+1}, R^X_{t+1}) + \mathbb{E}_t[m_{t+1}]\mathbb{E}_t[R^X_{t+1}] \end{aligned}$$

  which implies

$$\mathbb{E}_t[R^X_{t+1}] = \frac{1 - cov_t(m_{t+1}, R^X_{t+1})}{\mathbb{E}_t[m_{t+1}]} = R^{RF}_{t+1}\left[1 - cov_t(m_{t+1}, R^X_{t+1})\right]$$

- **Risk premium**

$$\text{risk premium} = \mathbb{E}_t[R^X_{t+1}] - R^{RF}_{t+1} \approx \frac{\mathbb{E}_t[R^X_{t+1}]}{R^{RF}_{t+1}} - 1 = -cov_t(m_{t+1}, R^X_{t+1})$$

  > 0 if m_{t+1} and R^X_{t+1} are negatively correlated

The equity premium puzzle

- Assume CRRA instantaneous utility for the representative household

$$u(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma}, \quad \sigma > 0$$

then, if the consumption grows at a random rate $g_{t+1} = c_{t+1}/c_t$

$$m_{t+1} = \frac{\beta u'(c_{t+1})}{u'(c_t)} = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} = \beta g_{t+1}^{-\sigma}$$

- Risk-free return is $R_{t+1}^{RF} = (\mathbb{E}_t[\beta g_{t+1}^{-\sigma}])^{-1}$
- Risk premium is approximately

$$-cov_t(\beta g_{t+1}^{-\sigma}, R_{t+1}^X)$$

> 0 if consumption growth g_{t+1} and asset return R_{t+1}^X are positively correlated

The equity premium puzzle

- It can be shown that if gross consumption growth g_{t+1} and gross equity return R^X_{t+1} are jointly log-normally distributed

$$\text{risk premium} \approx \ln(\mathbb{E}_t[R^X_{t+1}]) - \ln(R^{RF}_{t+1}) = \sigma \times cov_t(\ln g_{t+1}, \ln R^X_{t+1})$$

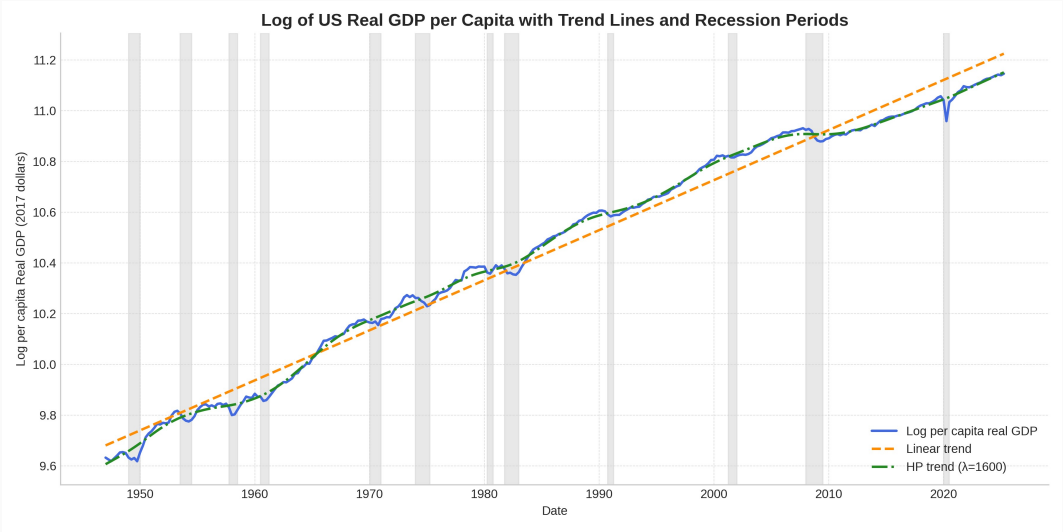
- However, in the data, consumption path is very smooth! Consumption growth volatility (SD) is about $1 \sim 2\%$
- Equity return (S&P 500) volatility (SD) is about $16 \sim 18\%$
- Even with perfectly correlated consumption growth and equity return, $cov_t(\ln g_{t+1}, \ln R^X_{t+1}) = 0.02 \times 0.18$, we still need $\sigma \approx 17 \sim 19$ to generate a $6 \sim 7\%$ equity premium
- But evidence suggests σ is close to 1 or 2, at least below 10...
- A relatively high σ (say 10 or 15) also makes it hard to match risk-free rate with reasonable β

Solutions to the puzzle?

- Separating risk-aversion from intertemporal substitution (Epstein-Zin preferences)
- Habit formation
- Rare disasters
- Taxes and distortions
- Heterogeneous agents and incomplete markets
- Financial constraints
- ...

Real Business Cycles (RBC) (next semester)

Time-series data and filters



Time-series data and filters

- **Business cycles**: fluctuations of the economy around its long-run growth trend
- Why do we usually plot macroeconomic variables (RGDP, consumption, investment etc.) in natural logs?
  - exponential growth = a straight line, growth rate = slope
  - log-difference $\approx$ percentage/proportional deviation
- Why do we usually use *seasonally adjusted* time-series data?
  - for higher-frequency (quarterly, monthly, etc.) data, some ups and downs are natural seasonal variations, and seasonal adjustment removes these
- How to separate these fluctuations from the trend of macroeconomic variables?

$$\underbrace{x_t}_{\text{observed}} = x_t^{trend} + x_t^{cyclical}$$

e.g., $x_t = \ln(\text{RGDP}_t)$.

The Hodrick-Prescott (HP) filter

- Most widely used method in macro to separate cyclical components and trends
- Given a time series $\{x_t\}_{t=0}^T$, the HP-filter finds the trend $\{x_t^{trend}\}_{t=0}^T$ by solving

$$\min_{\{x_t^{trend}\}_{t=0}^T} \left\{ \sum_{t=0}^{T} (x_t - x_t^{trend})^2 + \lambda \sum_{t=1}^{T-1} [(x_{t+1}^{trend} - x_t^{trend}) - (x_t^{trend} - x_{t-1}^{trend})]^2 \right\}$$

  trading off between
  - how closely the trend captures the volatility in data
  - how smooth the trend is
- Smoothing parameter λ governs this tradeoff
  - $\lambda \to 0$: trend = data
  - $\lambda \to \infty$: a smooth linear trend

The Hodrick-Prescott (HP) filter

- So lower-frequency data needs a smaller λ in general
- But choosing the right λ is to take a stand on what is the trend. It is an art...
  - quarterly data: $\lambda = 1600$ (as argued by Prescott (1986))
  - annual data: $\lambda = 100$ (old standard) or 6.25 (Ravn and Uhlig (2002))
  - monthly data: $\lambda = 14400$ (old standard) or 129600 (Ravn and Uhlig (2002))
- With the trend $\{x_t^{trend}\}_{t=0}^{T}$ at hand, cyclical components are $x_t^{cyclical} = x_t - x_t^{trend}$
- Drawbacks of the HP-filter include lack of theoretical supports, questionable end-points performance etc.
- Alternative filters: first difference, band-pass filter, Hamilton (2018), etc.

Stylized business-cycle facts (?)

The following observations are based on HP-filtered quarterly US data in logs

Volatilities

- *Output* (real GDP per capita) fluctuates around the trend with avg. SD $\approx 1.6\%$
- *Investment* is more volatile than output (2.5–3x)
- *Consumption* is less volatile than output (0.5–1x)
- *Employment* and *hours worked* are comparable to output in volatility (1–1.7x)

Stylized business-cycle facts (?)

Comovements and cyclicalities

- Investment, consumption, hours worked, employment, productivity are strongly *pro-cyclical*
  - a variable x_t is pro-cyclical if $corr(x_t, output_t) > 0$
- Unemployment rate is *counter-cyclical*
  - a variable x_t is counter-cyclical if $corr(x_t, output_t) < 0$

Persistence

- Macro time series exhibit significantly positive serial autocorrelation

Stylized business-cycle facts (?)

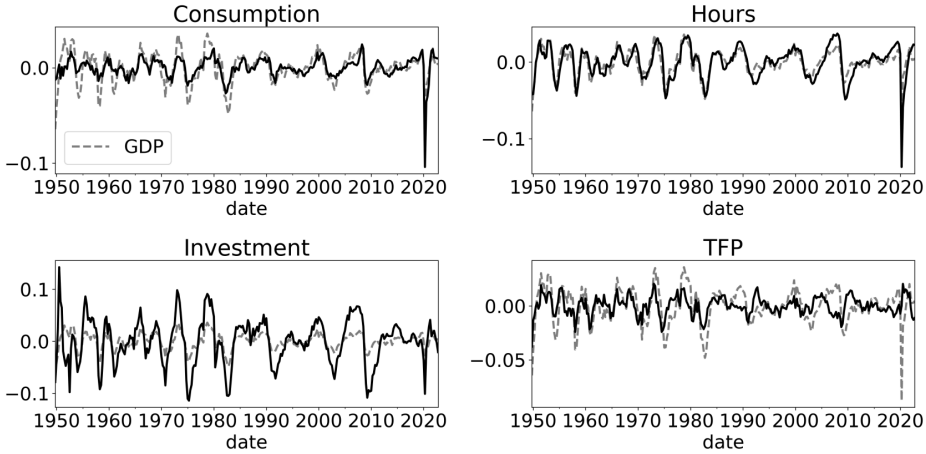


Figure 14.6: Deviations from Trend of U.S. Aggregates

Stylized business-cycle facts (?)

Table 14.2: Business cycle moments U.S. Data 1985-2022

Variable x	Standard Deviation (%)	Relative Std to σ_y	Auto-correlation	Cross correlation of x with $y(t-1)$	$y(t)$	$y(t+1)$
Output (y)	1.24	1.00	0.64	0.64	1.00	0.64
Consumption	1.20	0.97	0.56	0.56	0.86	0.43
Gov. Consumption	1.26	1.02	0.83	-0.32	-0.38	-0.39
Investment	3.64	2.94	0.90	0.63	0.79	0.70
Employment	1.56	1.26	0.72	0.69	0.78	0.34
Hours	2.19	1.77	0.74	0.70	0.88	0.52
Unemployment	15.60	12.59	0.74	-0.72	-0.83	-0.43
Lab. Productivity	1.09	0.88	0.76	-0.47	-0.29	-0.05
Wages	1.46	1.18	0.69	-0.21	-0.31	-0.06
Price Level	0.68	0.55	0.93	0.37	0.30	0.14
TFP	0.73	0.59	0.78	-0.33	-0.03	0.10
Fed Funds Rate	2.83	2.29	0.99	0.30	0.29	0.25

The real business cycle (RBC) model

- Baseline RBC model = stochastic NGM with endogenous labor supply
- Business-cycle fluctuations appear due to
  - Exogenous technology "shocks"
  - Endogenous response in aggregates to shocks
- Typical emphasis of business-cycle studies: endogenous
  - amplification
  - propagation (transmission + prolongation)

  of exogenous shocks

Setup

A simple baseline RBC model

- Same setup as the stochastic NGM model, but with endogenous labor supply
- Lifetime utility of the representative household (dropping event histories)

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t u(c_t, 1-l_t)\right], \quad \beta \in (0,1)$$

  where c_t is consumption at t, and l_t is labor supply at t, and 1 is fixed labor endowment.

- Assume u is strictly increasing and strictly concave and satisfies the Inada conditions in both arguments.

Setup

- A reasonable choice of $u(c_t, 1-l_t)$ consistent with a BGP

$$u(c_t, 1-l_t) = \frac{(c_t v(1-l_t))^{1-\sigma} - 1}{1-\sigma}, \quad \sigma > 0$$

with commonly used special cases:

$$u(c_t, 1-l_t) = \begin{cases} \ln c_t + \psi \ln(1-l_t) & \text{if } \sigma = 1,\ v(1-l_t) = (1-l_t)^{\psi}, \\ \ln c_t - \psi \dfrac{l_t^{1+1/\gamma}}{1+1/\gamma} & \text{if } \sigma = 1,\ v(1-l_t) = \exp(-\psi \frac{l_t^{1+1/\gamma}}{1+1/\gamma}),\ \gamma > 0 \\ \dfrac{(c_t(1-l_t)^{\psi})^{1-\sigma} - 1}{1-\sigma} & \text{if } \sigma \neq 1,\ v(1-l_t) = (1-l_t)^{\psi} \end{cases}$$

Social planner's problem in recursive formulation

- State variables: k, z
- Control variables: k', l
- Bellman equation

$$V(k,z) = \max_{k' \geq 0,\ l \in [0,1]} \left\{ u(c, 1-l) + \beta \sum_{z' \in Z} \pi(z' \mid z) V(k', z') \right\}$$
$$\text{where} \quad c = zF(k,l) + (1-\delta)k - k'$$

with the associated policy function $k' = g^K(k,z)$ and $l = g^L(k,z)$

Characterization of efficient allocation

- *Intra*-temporal optimal condition for labor supply at $k' = g^K(k,z)$ and $l = g^L(k,z)$

$$\underbrace{u_2(c,1-l)}_{\substack{\text{MC of supplying} \\ \text{1 more unit of labor}}} = \underbrace{u_1(c,1-l)}_{\substack{\text{MU of consuming} \\ \text{1 more unit}}} \underbrace{zF_2(k,l)}_{\text{MPL}} \iff MRS = \frac{u_2(c,1-l)}{u_1(c,1-l)} = zF_2(k,l)$$

- *Inter*-temporal Euler equation for capital stock at $k' = g^K(k,z)$ and $l = g^L(k,z)$

$$\begin{aligned} u_1(c,1-l) =& \beta \mathbb{E}[V_1(k',z') \mid z] = \beta \sum_{z' \in Z} \pi(z' \mid z) V_1(k',z') \\ =& \beta \mathbb{E}[u_1(c',1-l')[z'F_1(k',l') + 1 - \delta] \mid z] \end{aligned}$$

where $l' = g^L(k',z')$ is a random variable, and so is $c' = z'F(k',l') + (1-\delta)k' - k''$ where $k'' = g^K(k',z')$

(Numerical) solution

- Global solution (e.g., VFI) is less used in practice unless shocks are big
- Local solution (e.g., (log-)linearization or higher-order perturbation) is often sufficient
- Welfare theorems hold! So we can solve the planner's problem for allocations and then back out the prices.

Sequential markets and household's Bellman equation

- Now consider the market outcome: perfect competition, sequential trading
- Same as we did in the baseline stochastic NGM model
- State variables: k, z, K
- Pricing functions: $r(z, K), w(z, K)$
- Law of motion of aggregate capital: $K' = G(z, K)$
- Bellman equation

$$V(k, z, K) = \max_{k' \geq 0,\ l \in [0,1]} \left\{ u(c, 1-l) + \beta \sum_{z' \in Z} \pi(z' \mid z) V(k', z', G(z, K)) \right\}$$
$$\text{where} \quad c = w(z, K) l + (r(z, K) + 1 - \delta) k - k' \geq 0$$

with the associated policy functions $k' = g^k(k, z, K)$ and $l = g^l(k, z, K)$

Recursive competitive equilibrium

Definition 6

A **recursive competitive equilibrium** consists of household value function $V : \mathbb{R}^3_+ \to \mathbb{R}$ and policy functions $g^k, g^l : \mathbb{R}^3_+ \to \mathbb{R}_+$, **pricing functions** $r, w : \mathbb{R}^2_+ \to \mathbb{R}_+$, aggregate labor $L : \mathbb{R}^2_+ \to [0, 1]$, and the aggregate law of motion $G : \mathbb{R}^2_+ \to \mathbb{R}_+$ such that

- (household optimization) given r, w and G, the solution to the household's Bellman equation is V with an associated policy g^k, g^l;
- (firm optimization) r, w satisfy $r(z, K) = zF_1(K, L(z, K))$ and $w(z, K) = zF_2(K, L(z, K))$, $\forall z, K$;
- (**consistency**) individual decision is consistent with aggregate law of motion

$$g^k(K, z, K) = G(z, K),\ g^l(K, z, K) = L(z, K)\ \forall (z, K).$$

Quantitative performance

Question: to what extent do TFP shocks account for business-cycle fluctuations?

- Pick functional forms and parameter values consistent with long-run trend and micro evidence
  - calibrate most parameters at steady state/BGP
  - recall: calibration in deterministic NGM
- Estimate the TFP process
  - recall: TFP as Solow residuals
- Simulate the model to see whether it can replicate business-cycle moments

Calibration (no growth)

- Production function $F(K, L) = K^{\alpha} L^{1-\alpha}$ with $\alpha = 1/3$
- Quarterly frequency: common choice of $\beta = 0.99$, $\delta = 0.025$
  - at DSS, $\beta \underbrace{(1 - \delta + r)}_{=R} = 1$ with annual real interest rate at 4%, and $r = \frac{\alpha y}{k} = \alpha \delta \frac{y}{i}$
- Per-period utility function

$$u(c, 1 - l) = \ln c + \psi \ln(1 - l), \quad \psi > 0$$

  where ψ is picked to match $l = 1/3$ (8 hours a day):

$$\text{FOC at } z = 1: \ \frac{F_2}{c} = \frac{\psi}{1 - l} \implies \psi = \frac{1 - l}{l} \frac{1 - \alpha}{c/y} \approx \frac{2/3}{1/3} \frac{2/3}{0.8} \approx 1.7$$

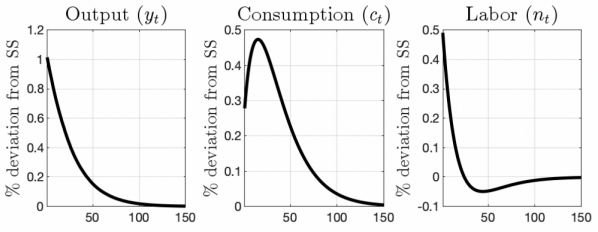
- Estimated AR(1) process for $\ln z_t$: $\rho_z = 0.95$, $\sigma_z = 0.007$

Simulated moments

Table 14.4: Business cycle moments in core RBC model

Variable x	Standard Deviation	Relative Std. Dev. σ_y	Auto-Correlation	Cross correlation of x with y
Output y	1.33	1	0.73	1
Consumption c	0.42	0.32	0.81	0.90
Investment i	4.13	3.10	0.72	0.99
Hours n	0.64	0.48	0.72	0.98
TFP z	0.92	0.69	0.43	0.99

Impulse response functions



- A 1-SD shock in TFP: $z_t \uparrow$
- On impact: labor $l_t \uparrow$ (due to dominant substitution effect) so output $y_t \uparrow$ more than z_t (**amplification**)
- Capital accumulation leads to lasting impact of the shock (**prolongation** or **propagation**)
- Small and sluggish adjustment in consumption due to consumption smoothing incentives

Readings

- AKMM Ch.7, Ch.11, Ch.14
- Krueger Ch.2, Ch.6

Natural debt limit with uncertainties

go back

- As in the deterministic case, a natural debt limit rules out the Ponzi scheme
- It never binds: a "set it and forget it" constraint
- With uncertainties, agent i's natural debt limit can be written as

$$\overline{A}_t^i(s^t) = \sum_{\tau=0}^{\infty} \sum_{s^{t+\tau}|s^t} \frac{p_{t+\tau}(s^{t+\tau})}{p_t(s^t)} e_{t+\tau}^i(s^{t+\tau})$$

where $p_t(s^t)$ is the Arrow-Debreu price for t, s^t. Intuition: agent i's indebtedness at t, s^t cannot exceed the value of their lifetime income henceforth

Natural debt limit in recursive form

go back

- In recursive form, the natural debt limit satisfies

$$\overline{A}^i(s, A) = e^i(s) + \sum_{s' \in S} q(s' \mid s, A)\, \overline{A}^i\big(s', G(s, A, s')\big)$$

- Intuition: the most agent i can owe at s is what they can repay with $c = 0$: today's endowment plus the most they can borrow against each s' at price $q(s' \mid s, A)$
- With complete markets, A drops out. With $Q := [q(s' \mid s)]_{s,s'}$,

$$\overline{A}^i(s) = e^i(s) + \sum_{s' \in S} q(s' \mid s)\, \overline{A}^i(s') \iff \overline{A}^i = (I - Q)^{-1} e^i = \sum_{\tau=0}^{\infty} Q^\tau e^i$$

the recursive version of the sequence formula sequence form, since $p_{t+\tau}/p_t$ is a product of q's