ECON 720: Advanced Macroeconomic Theory I

Uncertainty and Stochastic Neoclassical Growth

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Overview

- So far: deterministic models. Everyone knows everything in every period.
- But future is full of uncertainties.
- How to represent uncertainty/risks in dynamic macro models? Stochastic processes.
- Choices under uncertainty?
- Equilibria with uncertainty?

Overview

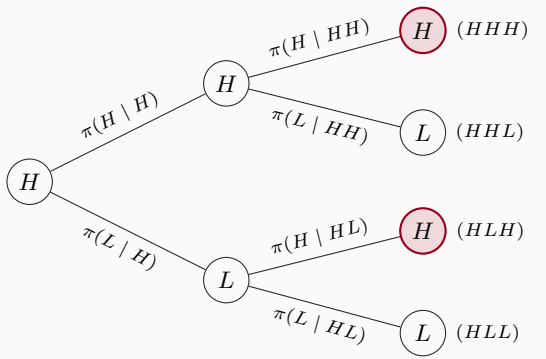
- Basic representation of uncertainty
- Stochastic growth: NGM + TFP shocks
  - Optimal growth and stochastic dynamic programming
  - Arrow-Debreu
  - Sequential markets and RCE
- Beyond Markov chains (FYI)
- Local vs global solution methods (optional)

Basics of Uncertainty

Event history

- Discrete time $t = 0, 1, 2, \ldots$
- Fixed, finite set of possible outcomes $S = \{\omega_1, \omega_2, \ldots, \omega_N\}$
- An **event** at time t: $s_t \in S$
- No uncertainty at $t = 0$: known and fixed $s_0 \in S$
- An **event history** up to time t: $s^t = (s_0, s_1, \ldots, s_t)$
- Set of all possible histories up to time t: $S^t = \{s_0\} \times \underbrace{S \times \ldots \times S}_{t}$
- Unconditional probability of observing a history s^t: $\pi_t(s^t)$, s.t. $\sum_{s^t \in S^t} \pi_t(s^t) = 1$
- A state of the economy at $t =$ an event history s^t, not just the current event s_t

Why index by histories?



- $s_t \in \{H, L\}$, $s_0 = H$ given: 2^t histories at t
- Each link: a conditional probability $\pi(s_{t+1} \mid s^t)$. Probability of a history: the product along its path, e.g.,

  $$\pi_2(HLH) = \pi(L \mid H)\,\pi(H \mid HL)$$

- At $t = 2$, $s_2 = H$ in both (HHH) and (HLH), but they are different states: the past differs, and so may everything that responded to it

Special cases

- Conditional probability of s^t given s^τ, $\tau < t$: $\pi(s^t \mid s^\tau)$
- Of particular interest: $\pi(s_{t+1} \mid s^t) = \pi(s^{t+1} \mid s^t)$
- Independent and identically distributed (iid) processes

$$\pi(s_{t+1} \mid s^t) = \pi(s_{t+1}), \quad \forall t$$

- Markov processes

$$\pi(s_{t+1} \mid s^t) = \pi(s_{t+1} \mid s_t), \quad \forall t$$

- These two cases are particularly convenient for recursive formulations

Special case: Markov chains

- Most practically/computationally relevant case
- A (discrete-time, discrete-state, time-homogenous) **Markov chain** includes a state space $S = \{\omega_1, \ldots, \omega_N\}$ and an $N \times N$ transition matrix P such that

$$P_{ij} = \Pr(X_{t+1} = \omega_j \mid X_t = \omega_i), \quad \forall i, j$$

  so $\sum_j P_{ij} = 1$. Let the initial distribution over S be θ_0 ($N \times 1$ vector) s.t. $X_0 \sim \theta_0$
- Conditional expectation of any function $v : S \to \mathbb{R}$ given $X_t = \omega_i$

$$\mathbb{E}[v(X_{t+1}) \mid X_t = \omega_i] = \sum_j P_{ij} v(\omega_j)$$

- Evolution of unconditional distribution $\theta_{t+1} = P^\top \theta_t$, a stationary distribution θ satisfies $\theta = P^\top \theta$

Special case: Markov chains

Example: 2 states, 1=good, 2=bad

- generic

$$P = \begin{pmatrix} p_{11} & 1-p_{11} \\ 1-p_{22} & p_{22} \end{pmatrix}$$

stationary distribution $\theta_1 = \dfrac{1-p_{22}}{(1-p_{11})+(1-p_{22})}$, $\theta_2 = \dfrac{1-p_{11}}{(1-p_{11})+(1-p_{22})}$

- symmetric

$$P = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

stationary distribution $\theta_1 = \theta_2 = 1/2$

- iid

$$P = \begin{pmatrix} p & 1-p \\ p & 1-p \end{pmatrix}$$

stationary distribution $\theta_1 = p$, $\theta_2 = 1-p$

Special case: AR(1) processes

- An autoregressive process of order one, i.e., AR(1), is a Markov process with a *continuous* state space (typically $\mathbb{R}$)

$$X_{t+1} = (1-\rho)\mu + \rho X_t + \epsilon_{t+1}, \quad \epsilon_{t+1} \overset{iid}{\sim} (0, \sigma^2)$$

- ρ measures the persistence of shocks
- If $\rho \in (-1, 1)$, the process is stationary with $\mathbb{E}[X_t] = \mu$ and $var(X_t) = \dfrac{\sigma^2}{1-\rho^2}$
- If $\rho = 1$, random walk, no well-defined unconditional mean or variance, $\mathbb{E}[X_{t+1} \mid X_t] = X_t$
- AR(1) is a very common formulation in macro models. We often discretize it into a Markov chain in computation.

Expected utility

- With uncertainty, an allocation becomes random
- Expected lifetime utility at time 0 before any realization of risks (without labor disutility)

$$U(\{\{c_t(s^t)\}_{s^t\in S^t}\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty}\sum_{s^t\in S^t}\beta^t\pi_t(s^t)u(c_t(s^t)) := \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t u(c_t)\right]$$

  where we enumerate all possible histories at all times

  - $c_t(s^t)$: consumption amount at t when the economy is at s^t
  - $\mathbb{E}_0[\cdot]$: expectation operator at $t=0$

- Concave $u \implies$ risk aversion: $\mathbb{E}[u(c)] \leq u(\mathbb{E}[c])$ by Jensen, the same inequality behind consumption smoothing (deck 4). CRRA: relative risk aversion $= \sigma = 1/\text{EIS}$ (deck 4, appendix)

Stochastic Neoclassical Growth

Setup

Identical setup to the simple deterministic neoclassical growth model

- $t = 0, 1, \ldots$
- a representative household and a representative firm
- consumption good, capital service, labor service each period

Plus aggregate productivity (TFP) as the only source of uncertainty

- production function at t

  $$z_t F(K, L)$$

  with the usual assumptions on F, but aggregate TFP follows a **Markov chain** over a finite Z with transition probabilities $\pi(z' \mid z)$
- history $z^t = (z_0, z_1, \ldots, z_t)$ with z_0 given (no uncertainty at $t = 0$)

Note: We mostly stick to Markov chains to avoid discussing measure theory related issues.

Setup

- Household expected lifetime utility

$$U(\{\{c_t(z^t)\}_{z^t \in Z^t}\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi_t(z^t) u(c_t(z^t)) := \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

- Capital accumulation

$$k_{t+1} = (1-\delta)k_t + i_t$$

  or, more accurately,

$$k_{t+1}(z^t) = (1-\delta)k_t(z^{t-1}) + i_t(z^t), \quad \forall t, z^t$$

  with k_0 given

- Household supplies 1 unit of labor inelastically

Plan

- We start from the planner's problem, i.e., optimal growth
  - Sequential formulation
  - Recursive formulation
- We move on to competitive equilibrium growth
  - Arrow-Debreu market structure
  - Sequential markets
  - Recursive competitive equilibrium

Optimal Growth: Sequential Formulation

Feasibility and Pareto efficiency

Definition 1 (Feasibility)
An allocation $\{\{c_t(z^t), i_t(z^t), k_{t+1}(z^t), l_t(z^t)\}_{z^t \in Z^t}\}_{t=0}^{\infty}$ is **feasible** if $\forall t,\ z^t = (z^{t-1}, z_t) \in Z^t$:

$$\begin{aligned} c_t(z^t) + i_t(z^t) \leq & z_t F(k_t(z^{t-1}), l_t(z^t)), \\ k_{t+1}(z^t) = & (1-\delta)k_t(z^{t-1}) + i_t(z^t), \end{aligned}$$

and $c_t(z^t), k_{t+1}(z^t) \geq 0$ with k_0 given, and $l_t(z^t) \in [0,1]$.

Definition 2 (Pareto efficiency)
An allocation $\{\{c_t(z^t), i_t(z^t), k_{t+1}(z^t), l_t(z^t)\}_{z^t \in Z^t}\}_{t=0}^{\infty}$ is **Pareto efficient** if it is feasible and there is no other feasible allocation $\{\{\widehat{c}_t(z^t), \widehat{i}_t(z^t), \widehat{k}_{t+1}(z^t), \widehat{l}_t(z^t)\}_{z^t \in Z^t}\}_{t=0}^{\infty}$ such that (identical) households are strictly better off, or

$$\sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi_t(z^t) u(\widehat{c}_t(z^t)) > \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi_t(z^t) u(c_t(z^t)).$$

Social planner's problem

The planner's problem in sequential formulation can be written as

$$
\begin{aligned}
W(k_0, z_0) = \max_{\{\{c_t(z^t), k_{t+1}(z^t)\}_{z^t \in Z^t}\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi_t(z^t) u(c_t(z^t)) \\
\text{subject to} \quad & c_t(z^t) + k_{t+1}(z^t) \leq z_t f(k_t(z^{t-1})) + (1-\delta) k_t(z^{t-1}), \ \forall t, z^t = (z^{t-1}, z_t) \\
& c_t(z^t), k_{t+1}(z^t) \geq 0, \ \forall t, z^t, \quad \text{given } k_0,
\end{aligned}
$$

where

- $W(k_0, z_0)$ is the value function
- We already plug in $l_t(z^t) = 1$ and $F(k, 1) = f(k)$
- redundant non-negativity constraints: Inada conditions imply $c_t(z^t) > 0$; and for any $c_{t+1}(z^t, z_{t+1}) > 0$, we need $k_{t+1}(z^t) > 0$, $\forall t, z^t, z_{t+1}$

Characterization

- Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \left[\beta^t \pi_t(z^t) u(c_t(z^t)) + \lambda_t(z^t)\big(z_t f(k_t(z^{t-1})) + (1-\delta)k_t(z^{t-1}) - c_t(z^t) - k_{t+1}(z^t)\big)\right]$$

- FOCs

$$\begin{aligned} \partial c_t(z^t): \quad & \beta^t \pi_t(z^t) u'(c_t(z^t)) = \lambda_t(z^t) \\ \partial k_{t+1}(z^t): \quad & \lambda_t(z^t) = \sum_{z_{t+1} \in Z} \lambda_{t+1}(z^t, z_{t+1})[z_{t+1} f'(k_{t+1}(z^t)) + 1 - \delta] \end{aligned}$$

Note how $k_{t+1}(z^t)$ appears in every budget constraint of $t+1$ when $z^{t+1} = (z^t, z_{t+1})$

Characterization: the Euler equations

- We derive the (stochastic) Euler equations

$$u'(c_t(z^t)) = \beta \sum_{z_{t+1} \in Z} \underbrace{\frac{\pi_{t+1}(z^t, z_{t+1})}{\pi_t(z^t)}}_{=\pi(z_{t+1}|z_t)} u'(c_{t+1}(z^t, z_{t+1}))[z_{t+1} f'(k_{t+1}(z^t)) + 1 - \delta]$$

- Writing more concisely

$$u'(c_t(z^t)) = \beta \mathbb{E}\left[u'(c_{t+1}(z^t, z_{t+1}))[z_{t+1} f'(k_{t+1}(z^t)) + 1 - \delta] \mid z_t\right], \quad \forall t, z^t$$

  or, dropping history when not confusing to ease notation:

$$u'(c_t) = \beta \mathbb{E}_t\left[u'(c_{t+1})[z_{t+1} f'(k_{t+1}) + 1 - \delta]\right]$$

  where $\mathbb{E}_t[\cdot]$ is the conditional expectation operator

- Interpretation: MC at t = *expected* MB at $t+1$

Characterization: the expected TVC

- As in the deterministic case, we need a transversality condition to ensure optimality:

$$\lim_{t\to\infty} \mathbb{E}_0 \left[\beta^t u'(c_t(z^t))[z_t f'(k_t(z^{t-1})) + 1 - \delta]k_t(z^{t-1})\right] = 0$$

  or equivalently

$$\lim_{t\to\infty} \mathbb{E}_0 \left[\beta^t u'(c_t(z^t))k_{t+1}(z^t)\right] = 0$$

- If an allocation sequence $\{\{c_t^*(z^t), k_{t+1}^*(z^t)\}_{z^t\in Z^t}\}_{t=0}^{\infty}$ satisfies the Euler equations and the TVC, it solves the planner's problem.
- Much more convenient to work with the recursive formulation!

Optimal Growth: Recursive Formulation

Planner's problem in recursive formulation

- Dynamic programming to the rescue: most of the deterministic DP results go through
- State variables for the planner:

$$k, z$$

  k: current-period capital stock, z: current-period productivity

- z must be a state because
  - need to know z to know total resource available $zf(k) + (1 - \delta)k$
  - Markov: distribution of next period's z' depends on z, $\pi(z' \mid z)$

Planner's Bellman equation

Planner's Bellman equation similar to deterministic case except for an expectation operator:

$$V(k,z) = \max_{k' \in [0, zf(k)+(1-\delta)k]} \left\{ u(zf(k) + (1-\delta)k - k') + \beta \underbrace{\sum_{z' \in Z} \pi(z' \mid z) V(k', z')}_{=\mathbb{E}[V(k',z')|z]} \right\}$$

The associated policy function is denoted as $k' = g(k, z)$

Characterization: the functional Euler equation

- FOC wrt k' evaluated at $k' = g(k, z)$

$$u'(zf(k) + (1-\delta)k - k') = \beta \mathbb{E}[V_1(k', z') \mid z]$$

- Envelope condition

$$V_1(k, z) = u'(zf(k) + (1-\delta)k - k')[zf'(k) + 1 - \delta]$$

- Functional Euler equation

$$u'(zf(k) + (1-\delta)k - k') = \beta \mathbb{E}[u'(z'f(k') + (1-\delta)k' - k'')[z'f'(k') + 1 - \delta] \mid z]$$

where k'' is a random variable such that $k'' = g(k', z') = g(g(k, z), z')$

- How to solve the model? Exact solution in special cases, or VFI local methods (optional)

An analytical example: exact solution

- $u(c) = \ln c$, $f(k) = k^\alpha$, $\delta = 1$, and productivity follows a Markov process $\pi(z' \mid z)$
- Planner's problem becomes

$$V(k, z) = \max_{k' \in [0, zk^\alpha]} \{\ln(zk^\alpha - k') + \beta \mathbb{E}[V(k', z') \mid z]\}$$

- Euler equation becomes

$$\frac{1}{zk^\alpha - k'} = \beta \mathbb{E}\left[\frac{\alpha z' k'^{\alpha-1}}{z' k'^\alpha - k''} \mid z\right]$$

- Guess a *constant saving rate* $s = k'/zk^\alpha$, i.e., a policy function $g(k, z) = szk^\alpha$:

$$\frac{s}{1-s} = \alpha\beta \mathbb{E}\left[\frac{1}{1-s} \mid z\right] \quad \Longrightarrow \quad s = \alpha\beta$$

- If we also want the value function, we can further guess and verify that it takes the form $V(k, z) = A \ln k + B(z) + C$...

An analytical example: exact solution

$$\begin{aligned} V(k,z) =& A\ln k + B(z) + C \\ =& \ln(1-s) + \ln z + \alpha \ln k + \beta \mathbb{E}[A(\ln s + \ln z + \alpha \ln k) + B(z') + C \mid z] \end{aligned}$$

- So we have

$$\begin{aligned} A =& \alpha + \alpha\beta A = \frac{\alpha}{1-\alpha\beta} \\ B(z) =& (1+\beta A)\ln z + \beta \mathbb{E}[B(z') \mid z] = \frac{\ln z}{1-\alpha\beta} + \beta \mathbb{E}[B(z') \mid z] \\ C =& \ln(1-s) + \beta A \ln s + \beta C = \frac{1}{1-\beta}\left(\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta)\right) \end{aligned}$$

  where we know the function $B(z)$ exists and is unique when z is bounded as Blackwell's sufficient conditions hold.

- Special case AR(1): $\ln z_{t+1} = (1-\rho)\mu_z + \rho \ln z_t + \epsilon_{t+1}$, $\rho \in [0,1]$, $\epsilon_{t+1} \overset{iid}{\sim} (0, \sigma^2)$. Then $B(z) = \dfrac{\ln z + \frac{\beta(1-\rho)}{1-\beta}\mu_z}{(1-\beta\rho)(1-\alpha\beta)}$

What replaces the steady state?

- With shocks, (k, z) keeps moving: no steady state in the deterministic sense
- Three useful notions
  - **deterministic steady state (DSS)**: shut off the shocks ($\epsilon_t \equiv 0$), so $z_t \equiv \bar{z}$ forever: $1 = \beta[\bar{z} f'(k) + 1 - \delta]$[1]
  - **stochastic (risky) steady state (SSS)**: households expect shocks, but z happens to stay at $\bar{z}$: $k = g(k, \bar{z})$
  - **stationary distribution** of (k, z): the long-run distribution the economy keeps visiting[2]
- In general SSS $\neq$ DSS: risk changes saving incentives (e.g., precautionary saving, next deck)
- Analytical example: the saving rate $\alpha\beta$ does not depend on risk, so SSS = DSS. The economy still fluctuates around it

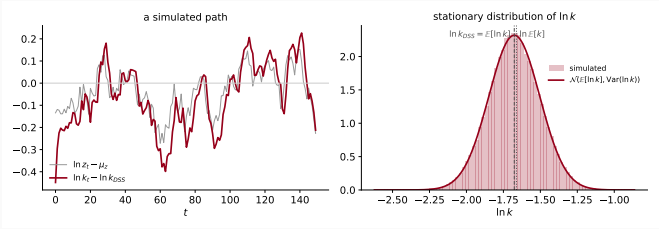
[1] With $\ln z$ an AR(1), $\ln \bar{z} = \mathbb{E}[\ln z]$, not $\ln \mathbb{E}[z]$. Not harmless: the gap is second order in σ, as large as the effects of risk.
[2] Here it is unique and globally stable, so one long simulation traces it out (ergodicity).

The analytical example: stationary distribution

- $\ln k' = \ln(\alpha\beta) + \ln z + \alpha \ln k$: $\ln k$ inherits the persistence of $\ln z$. With the AR(1) for $\ln z$, $\ln k$ is normal in the long run (if ϵ is), with

$$\mathbb{E}[\ln k] = \frac{\ln(\alpha\beta) + \mu_z}{1-\alpha} = \ln k_{DSS}, \qquad \text{Var}(\ln k) = \sigma_z^2 \frac{1+\alpha\rho}{(1-\alpha^2)(1-\alpha\rho)}$$

- But $\mathbb{E}[k] > k_{DSS}$ by Jensen: $\ln \mathbb{E}[k] = \mathbb{E}[\ln k] + \frac{1}{2}\text{Var}(\ln k)$
- Heterogeneous-agent models (next semester): the same object is the cross-sectional distribution



$\alpha = 0.36,\ \beta = 0.95,\ \rho = 0.9,\ \sigma_\epsilon = 0.05,\ \mu_z = 0$

Numerical global solution: value function iteration

Value function iteration still exploits the contraction mapping theorem. Consider a more general stochastic dynamic programming problem

$$V(x,z) = (TV)(x,z) := \max_{x' \in \Gamma(x,z)} \left\{ \mathcal{F}(x,x',z) + \beta \sum_{z' \in Z} \pi(z' \mid z) V(x',z') \right\}$$

Algorithm:

1. Initialization: select any function $V_0(x,z)$, $\forall (x,z)$ (e.g., $V_0(x,z) = 0$);
2. Update: given $V_n(x,z)$ from the previous iteration, update V_{n+1} as $V_{n+1} = TV_n$, i.e.,

$$V_{n+1}(x,z) = \max_{x' \in \Gamma(x,z)} \left\{ \mathcal{F}(x,x',z) + \beta \sum_{z' \in Z} \pi(z' \mid z) V_n(x',z') \right\}, \quad \forall (x,z)$$

3. Check convergence: if $\|V_{n+1} - V_n\| \leq \epsilon$, where ϵ is a pre-determined tolerance level, stop; otherwise, go back to step 2.

Arrow-Debreu Equilibrium

Time-0 trading

- All markets open and trades occur at $t = 0$ *before* any risk is realized
- Each commodity (consumption good, labor services, capital services) must be indexed by the **entire state history** z^t for any t
  - Why?
  - Prices: $\{\{\widehat{p}_t(z^t), \widehat{w}_t(z^t), \widehat{r}_t(z^t)\}_{z^t \in Z^t}\}_{t=0}^{\infty}$
  - e.g., with a finite horizon $t = 0, 1, \ldots, T$, no uncertainty at $t = 0$, $3\sum_{t=0}^{T} |Z|^t$ markets are open (one per commodity per history)
- Agents trade **state contingent** (i.e., history dependent) claims of each commodity
- All markets are competitive
- Households own firms

A representative firm's problem

- At $t = 0$, when all markets are open, the firm sees and **takes as given all prices** $\{\{\widehat{p}_t(z^t), \widehat{r}_t(z^t), \widehat{w}_t(z^t)\}\}_{t=0}^{\infty}$ to maximize profit

$$\widehat{\Pi} = \max_{\{K_t(z^t), L_t(z^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \left(\widehat{p}_t(z^t) z_t F(K_t(z^t), L_t(z^t)) - \widehat{r}_t(z^t) K_t(z^t) - \widehat{w}_t(z^t) L_t(z^t)\right)$$

  which is equivalent to solving a sequence of static problems, such that $\forall t$,

$$\widehat{\pi}_t(z^t) = \max_{K_t(z^t), L_t(z^t)} \widehat{p}_t(z^t) z_t F(K_t(z^t), L_t(z^t)) - \widehat{r}_t(z^t) K_t(z^t) - \widehat{w}_t(z^t) L_t(z^t)$$

  and $\widehat{\Pi} = \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \widehat{\pi}_t(z^t)$.

- No decisions are made from $t = 1$ onward

A representative household's problem

At $t = 0$, when all markets are open, the household with k_0 sees and **takes as given all prices** $\{\{\widehat{p}_t(z^t), \widehat{r}_t(z^t), \widehat{w}_t(z^t)\}\}_{t=0}^{\infty}$ and profit $\widehat{\Pi}$ to maximize lifetime utility

$$\max_{\{\{c_t(z^t), i_t(z^t), k_{t+1}(z^t)\}_{z^t \in Z^t}\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right] = \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi_t(z^t) u(c_t(z^t))$$

subject to

$$\sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \widehat{p}_t(z^t)(c_t(z^t) + i_t(z^t)) \leq \sum_{t=0}^{\infty} \sum_{z^t = (z^{t-1}, z_t) \in Z^t} (\widehat{r}_t(z^t) k_t(z^{t-1}) + \widehat{w}_t(z^t)) + \widehat{\Pi},$$

$$k_{t+1}(z^t) = (1-\delta) k_t(z^{t-1}) + i_t(z^t), \ \forall t, z^t = (z^{t-1}, z_t)$$

$$c_t(z^t) \geq 0, k_{t+1}(z^t) \geq 0, \ \forall t, z^t, \text{ given } k_0.$$

Some observations

- **Only 1 budget constraint (all trades at $t = 0$)**. No decisions after $t = 0$.
- No probabilities in the budget constraint! Only prices and quantities.
- **Only relative prices matter**: need a numeraire

Some observations

- Let time-0 consumption good be the numeraire: $\widehat{p}_0(z_0) = 1$. Write the sequence of relative good prices as $\{\{p_t(z^t)\}_{z^t \in Z^t}\}_{t=0}^{\infty}$, where $p_t(z^t) := \widehat{p}_t(z^t)/\widehat{p}_0(z_0)$: price of history-$z^t$ good at t in the unit of time-0 good.

- Rewrite history-z^t interest rate, wage, and profit in the unit of history-z^t good: $r_t(z^t) := \widehat{r}_t(z^t)/\widehat{p}_t(z^t)$ and $w_t(z^t) := \widehat{w}_t(z^t)/\widehat{p}_t(z^t)$, $\pi_t(z^t) := \widehat{\pi}_t(z^t)/\widehat{p}_t(z^t)$

- Then firm problem becomes the following sequence of static decisions: $\forall t, z^t$,

$$\max_{K_t(z^t), L_t(z^t) \geq 0} z_t F(K_t(z^t), L_t(z^t)) - r_t(z^t) K_t(z^t) - w_t(z^t) L_t(z^t)$$

and household budget constraint becomes ($\Pi := \widehat{\Pi}/\widehat{p}_0 = \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} p_t(z^t)\pi_t(z^t)$)

$$\sum_{t=0}^{\infty} \sum_{z^t \in Z^t} p_t(z^t)(c_t(z^t) + i_t(z^t)) \leq \sum_{t=0}^{\infty} \sum_{z^t = (z^{t-1}, z_t) \in Z^t} p_t(z^t)(r_t(z^t)k_t(z^{t-1}) + w_t(z^t)) + \Pi$$

Equilibrium definition

Definition 3
An **Arrow-Debreu equilibrium** consists of prices $\{\{p_t^*(z^t), p_t^*(z^t)r_t^*(z^t), p_t^*(z^t)w_t^*(z^t)\}_{z^t \in Z^t}\}_{t=0}^{\infty}$ with $p_0^*(z_0) = 1$, time-0 profit Π, household allocation $\{\{c_t^*(z^t), i_t^*(z^t), k_{t+1}^*(z^t)\}_{z^t \in Z^t}\}_{t=0}^{\infty}$ given k_0, and firm allocation $\{\{K_t^d(z^t), L_t^d(z^t)\}_{z^t \in Z^t}\}_{t=0}^{\infty}$, such that

- (Household optimization) given prices and Π, $\{\{c_t^*(z^t), i_t^*(z^t), k_{t+1}^*(z^t)\}_{z^t \in Z^t}\}_{t=0}^{\infty}$ solves the household's problem;
- (Firm optimization) given prices, $\{K_t^d(z^t), L_t^d(z^t)\}_{t=0}^{\infty}$ solves the firm's problem such that the optimized total profit evaluated at $t = 0$ is Π;
- (Market clearing) all markets opening at $t = 0$ clear:

$$c_t^*(z^t) + i_t^*(z^t) = z_t F(K_t^d(z^t), L_t^d(z^t)), \quad k_t^*(z^{t-1}) = K_t^d(z^t), \quad 1 = L_t^d(z^t), \quad \forall t, z^t.$$

Analysis of firm problem

- Given z^t-prices denominated in time-0 consumption good $p_t(z^t), p_t(z^t)w_t(z^t), p_t(z^t)r_t(z^t)$, $\forall t$

$$\pi_t(z^t) = \max_{K_t(z^t), L_t(z^t) \geq 0} z_t F(K_t(z^t), L_t(z^t)) - r_t(z^t)K_t(z^t) - w_t(z^t)L_t(z^t)$$

  with FOCs

$$z_t F_1(K_t(z^t), L_t(z^t)) = r_t(z^t), \qquad z_t F_2(K_t(z^t), L_t(z^t)) = w_t(z^t).$$

- CRS implies $\pi_t(z^t) = 0$, $\forall t$, so total profit at $t = 0$ is 0:

$$\Pi = \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} p_t(z^t)\pi_t(z^t) = 0$$

- Only K/L ratio can be pinned down b/c F_1 and F_2 are homogenous of degree 0

Analysis of household problem

- Replace $i_t(z^t) = k_{t+1}(z^t) - (1-\delta)k_t(z^{t-1})$
- Given price sequence $\{p_t(z^t), p_t(z^t)r_t(z^t), p_t(z^t)w_t(z^t)\}_{t=0}^{\infty}$ and $\Pi = 0$, rewrite and simplify household problem as

$$\max_{\{c_t(z^t),k_{t+1}(z^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi_t(z^t) u(c_t(z^t))$$

subject to

$$\sum_{t=0}^{\infty} \sum_{z^t=(z^{t-1},z_t)} p_t(z^t)\Big(c_t(z^t) + k_{t+1}(z^t) - (1-\delta)k_t(z^{t-1})\Big)$$

$$\leq \sum_{t=0}^{\infty} \sum_{z^t=(z^{t-1},z_t)} p_t(z^t)\Big(r_t(z^t)k_t(z^{t-1}) + w_t(z^t)\Big)$$

$$k_{t+1}(z^t) \geq 0, \ \forall t, \ \text{ given } k_0.$$

Analysis of household problem

- Let λ be the Lagrangian multiplier for the budget constraint (ignore $k_{t+1}(z^t) \geq 0$ by the same logic in the deterministic case)

$$\begin{aligned}\mathcal{L} =& \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi_t(z^t) u(c_t(z^t)) \\ &+ \lambda \Bigg[\sum_{t=0}^{\infty} \sum_{z^t = (z^{t-1}, z_t)} p_t(z^t) \Big(r_t(z^t) k_t(z^{t-1}) + w_t(z^t) \Big) \\ &\quad - \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} p_t(z^t) \Big(c_t(z^t) + k_{t+1}(z^t) - (1-\delta) k_t(z^{t-1}) \Big) \Bigg]\end{aligned}$$

- FOCs wrt $c_t(z^t)$ and $k_{t+1}(z^t)$ imply

$$\begin{aligned}\beta^t \pi_t(z^t) u'(c_t(z^t)) =& \lambda p_t(z^t) \\ \lambda \Bigg(p_t(z^t) - \sum_{z_{t+1} \in Z} p_{t+1}(z^t, z_{t+1}) (r_{t+1}(z^t, z_{t+1}) + 1 - \delta) \Bigg) =& 0\end{aligned}$$

Analysis of household problem

- Observe: FOC wrt k_{t+1} is essentially a restriction on price sequence:

$$p_t(z^t) = \sum_{z_{t+1} \in Z} p_{t+1}(z^t, z_{t+1})(r_{t+1}(z^t, z_{t+1}) + 1 - \delta)$$

- At $t = 0$, we obtain

$$\lambda = u'(c_0(z_0))$$

  Interpretation: λ is the marginal utility gain when the present value of wealth increases by one unit.

Analysis of household problem

- From the FOCs, we have the marginal rate of substitution

$$\frac{\beta \pi_{t+1}(z^{t+1}) u'(c_{t+1}(z^{t+1}))}{\pi_t(z^t) u'(c_t(z^t))} = \frac{p_{t+1}(z^{t+1})}{p_t(z^t)},$$

- the stochastic **Euler equations** for the household's decision

$$\begin{aligned} u'(c_t(z^t)) =& \beta \sum_{z_{t+1} \in Z} \underbrace{\frac{\pi_{t+1}(z^t, z_{t+1})}{\pi_t(z^t)}}_{=\pi(z_{t+1}|z_t)} u'(c_{t+1}(z^t, z_{t+1}))(r_{t+1}(z^t, z_{t+1}) + 1 - \delta) \\ =& \beta \mathbb{E}[u'(c_{t+1}(z^{t+1}))(r_{t+1}(z^{t+1}) + 1 - \delta) \mid z_t], \quad \forall t, z^t \end{aligned}$$

MC at z^t = expected MB at $z^{t+1} = (z^t, z_{t+1})$

Equilibrium characterization: allocation

- When all markets clear, we have $K_t^d(z^t) = k_t^*(z^{t-1})$ and $L_t^d(z^t) = 1$, so $F(k_t^*(z^{t-1}), 1) = f(k_t^*(z^{t-1}))$ and

$$c_t^*(z^t) = f(k_t^*(z^{t-1})) + (1-\delta)k_t^*(z^{t-1}) - k_{t+1}^*(z^t),\ \forall t$$

and

$$r_t^*(z^t) = z_t F_1(k_t^*(z^{t-1}), 1) = z_t f'(k_t^*(z^{t-1})),\ \forall t$$

- In equilibrium, the Euler equation must satisfy

$$u'(c_t^*(z^t)) = \beta \sum_{z_{t+1} \in Z} \pi(z_{t+1} \mid z_t) u'(c_{t+1}^*(z^{t+1}))(z_{t+1} f'(k_{t+1}^*(z^t)) + 1 - \delta), \quad \forall t$$

which coincides with the social planner's.

Equilibrium characterization: allocation

- Planner's TVC holds at the ADE

$$\lim_{t\to\infty} \sum_{z^t \in Z^t} \beta^t \pi_t(z^t) u'(c_t^*(z^t)) k_{t+1}^*(z^t) = \lambda \lim_{t\to\infty} \sum_{z^t \in Z^t} p_t^*(z^t) k_{t+1}^*(z^t) = 0$$

  where the second equality must hold for a well-defined Arrow-Debreu budget constraint.

- Therefore, the allocation at the **Arrow-Debreu equilibrium** coincides with what is **Pareto optimal**.

- Welfare theorems

Equilibrium characterization: prices

- ADE prices must satisfy

$$
\begin{aligned}
r_t^*(z^t) =& z_t F_1(k_t^*(z^{t-1}), 1) = z_t f'(k_t^*(z^{t-1})),\ \forall t \\
w_t^*(z^t) =& z_t F_2(k_t^*(z^{t-1}), 1) = z_t f(k_t^*(z^{t-1})) - k_t^*(z^{t-1}) z_t f'(k_t^*(z^{t-1})),\ \forall t \\
p_t^*(z^t) =& \frac{\beta^t \pi_t(z^t) u'(c_t^*(z^t))}{u'(c_0^*(z_0))},\ \forall t \geq 1
\end{aligned}
$$

- All markets must clear simultaneously: note how "everything affects everything"

- How do we think about $p_t^*(z^t)$? **A state-contingent intertemporal exchange rate or price of a history**: we need to give up $p_t^*(z^t)$ units of consumption good in 0 in exchange for 1 unit of consumption good delivered in z^t

Interpreting state prices

- Decompose:

$$p_t^*(z^t) = \underbrace{\beta^t}_{\text{discounting}} \times \underbrace{\pi_t(z^t)}_{\text{probability}} \times \underbrace{\frac{u'(c_t^*(z^t))}{u'(c_0^*(z_0))}}_{\text{scarcity}}$$

  Higher if z^t is more likely *and* if consumption is scarce at z^t (insurance value)

- One period ahead:

$$\frac{p_{t+1}^*(z^{t+1})}{p_t^*(z^t)} = \pi(z_{t+1} \mid z_t)\, m_{t+1}, \qquad m_{t+1} := \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

  m_{t+1}: the **stochastic discount factor** (more in deck 6)

Sequential Markets Equilibrium

Sequential trading

- Markets for history-z^t commodities open at history z^t, $\forall t, z^t$
  - history-z^t consumption good, with normalized price 1
  - history-z^t capital service, with price $r_t(z^t)$ in the unit of history-z^t good
  - history-z^t labor service, with price $w_t(z^t)$ in the unit of history-z^t good
- Why can we normalize the good price in every history to be 1, i.e., why do we have a numeraire every period?

Firm and household problems

- At every history z^t, given prices $r_t(z^t), w_t(z^t)$, firm maximizes per-period profit

$$\pi_t(z^t) = \max_{K_t(z^t), L_t(z^t) \geq 0} z_t F(K_t(z^t), L_t(z^t)) - r_t(z^t) K_t(z^t) - w_t(z^t) L_t(z^t)$$

- At $t = 0$, the household still maximizes the lifetime utility, knowing markets will open period by period, and taking as given all prices and profits $\{r_t(z^t), w_t(z^t), \pi_t(z^t)\}_{t=0}^{\infty}$

$$\begin{aligned}
&\max_{\{\{c_t(z^t), i_t(z^t), k_{t+1}(z^t)\}_{z^t \in Z^t}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi_t(z^t) u(c_t(z^t)) \\
&\text{subject to} \quad c_t(z^t) + i_t(z^t) \leq r_t(z^t) k_t(z^{t-1}) + w_t(z^t) + \pi_t(z^t), \ \forall t, z^t = (z^{t-1}, z_t) \\
&\qquad\qquad k_{t+1}(z^t) = (1-\delta) k_t(z^{t-1}) + i_t(z^t), \ \forall t, z^t = (z^{t-1}, z_t) \\
&\qquad\qquad c_t(z^t) \geq 0, \ \forall t \\
&\qquad\qquad k_{t+1}(z^t) \geq 0, \ \forall t, \ \text{given } k_0.
\end{aligned}$$

A budget constraint at every history z^t: expenditure cannot exceed income at any z^t.

Equilibrium definition

Definition 4
A **sequential markets equilibrium** consists of a sequence of prices $\{\{r_t^*(z^t), w_t^*(z^t)\}_{z^t \in Z^t}\}_{t=0}^{\infty}$, per-period profit $\{\{\pi_t^*(z^t)\}_{z^t \in Z^t}\}_{t=0}^{\infty}$, household allocation $\{\{c_t^*(z^t), i_t^*(z^t), k_{t+1}^*(z^t)\}_{z^t \in Z^t}\}_{t=0}^{\infty}$ given k_0, and firm allocation $\{\{K_t^d(z^t), L_t^d(z^t)\}_{z^t \in Z^t}\}_{t=0}^{\infty}$, such that

- (Household optimization) given prices and $\{\{\pi_t^*(z^t)\}_{z^t \in Z^t}\}_{t=0}^{\infty}$, $\{\{c_t^*(z^t), i_t^*(z^t), k_{t+1}^*(z^t)\}_{z^t \in Z^t}\}_{t=0}^{\infty}$ solves the household's problem;
- (Firm optimization) $\forall t, z^t$, given $r_t^*(z^t), w_t^*(z^t)$, the firm achieves maximal flow profit $\pi_t^*(z^t)$ at $K_t^d(z^t), L_t^d(z^t)$;
- (Market clearing) all markets clear at every t, z^t:

$$c_t^*(z^t) + i_t^*(z^t) = z_t F(K_t^d(z^t), L_t^d(z^t)), \qquad k_t^*(z^{t-1}) = K_t^d(z^t), \qquad 1 = L_t^d(z^t), \qquad \forall t.$$

Analysis of household problem

- Firm problem is identical. Omitted.
- Given $\{\{r_t(z^t), w_t(z^t)\}_{z^t \in Z^t}\}_{t=0}^{\infty}$ and $\pi_t(z^t) = 0$, $\forall t, z^t$, household problem can be written as

$$\max_{\{\{c_t(z^t), k_{t+1}(z^t)\}_{z^t \in Z^t}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi_t(z^t) u(c_t(z^t))$$

$$\text{subject to} \quad \begin{aligned} c_t(z^t) + k_{t+1}(z^t) &\leq (r_t(z^t) + 1 - \delta) k_t(z^{t-1}) + w_t(z^t), \ \forall t, z^t = (z^{t-1}, z_t) \\ k_{t+1}(z^t) &\geq 0, \ \forall t, \quad \text{given } k_0. \end{aligned}$$

- Let $\lambda_t(z^t)$ be the Lagrangian multiplier for the budget constraint in event history z^t at t (and ignore the constraint $k_{t+1}(z^t) \geq 0$)

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \Big(\beta^t \pi_t(z^t) u(c_t(z^t)) + \lambda_t(z^t) \Big[(r_t(z^t) + 1 - \delta) k_t(z^{t-1}) + w_t(z^t) - c_t(z^t) - k_{t+1}(z^t)$$

Analysis of household problem

- FOCs wrt $c_t(z^t)$ and $k_{t+1}(z^t)$ imply

$$\begin{aligned}\beta^t \pi_t(z^t) u'(c_t(z^t)) =& \lambda_t(z^t) \\ \lambda_t(z^t) =& \sum_{z_{t+1} \in Z} \lambda_{t+1}(z^t, z_{t+1})(r_{t+1}(z^t, z_{t+1}) + 1 - \delta)\end{aligned}$$

- So we have the Euler equations

$$u'(c_t(z^t)) = \sum_{z_{t+1} \in Z} \beta \pi(z_{t+1} \mid z_t) u'(c_{t+1}(z^t, z_{t+1}))(r_{t+1}(z^t, z_{t+1}) + 1 - \delta), \quad \forall t, z^t$$

- The transversality condition (TVC) now can be written as

$$\lim_{t \to \infty} \sum_{z^t \in Z^t} \lambda_t(z^t) k_{t+1}(z^t) = \lim_{t \to \infty} \sum_{z^t \in Z^t} \beta^t \pi_t(z^t) u'(c_t(z^t)) k_{t+1}(z^t) = 0$$

Equilibrium characterization

- It is straightforward to verify that SME outcome is once again efficient
  - Household's Euler equations and TVC evaluated at SME coincide with the planner's
- SME prices are equivalent to the ADE prices

$$
\begin{aligned}
r_t^*(z^t) =& z_t F_1(k_t^*(z^{t-1}), 1) = z_t f'(k_t^*(z^{t-1})) \\
w_t^*(z^t) =& z_t F_2(k_t^*(z^{t-1}), 1) = z_t f(k_t^*(z^{t-1})) - k_t^*(z^{t-1}) z_t f'(k_t^*(z^{t-1}))
\end{aligned}
$$

- Note: this SME has *no* state-contingent assets, yet it matches the ADE. Why?

Arrow securities and complete markets

- Add one-period **Arrow securities**: at z^t, buy $a_{t+1}(z^t, z')$ units at price $q_t(z^t, z')$, each paying 1 unit of good next period if and only if $z_{t+1} = z'$

$$c_t + k_{t+1} + \sum_{z' \in Z} q_t(z^t, z')\, a_{t+1}(z^t, z') \leq (r_t + 1 - \delta)k_t + w_t + a_t(z^t)$$

- $|Z|$ securities per history span all contingencies: markets are **complete**, and (with a no-Ponzi condition) the budget set is the same as AD's. Without uncertainty, one bond per period was enough (deck 4)
- FOC for $a_{t+1}(z^t, z')$: $q_t(z^t, z') = \beta\pi(z' \mid z_t)\dfrac{u'(c_{t+1}(z^t, z'))}{u'(c_t(z^t))} = \dfrac{p_{t+1}(z^t, z')}{p_t(z^t)}$
- Zero net supply and a representative household: $a^* = 0$. Prices exist, but nobody trades, so the SME without Arrow securities has the same allocation
- Preview (deck 6): with heterogeneous households, trades are nonzero (risk sharing); and q prices any asset (asset pricing)

Recursive Competitive Equilibrium

Aggregate state variables

- At any given t, z^t, what's the minimal set of information for household to know current prices and infer future prices?
- Aggregate state

$$z, K$$

- A time-invariant mapping from (z, K) to prices $r(z, K)$ and $w(z, K)$
- The law of motion for aggregate capital

$$K' = G(z, K)$$

- Transition probabilities

$$\pi(z' \mid z)$$

Individual state and Bellman equation

- Household also needs to know own k to know the budget constraint
- Control variable is still next-period's capital stock k'
- Bellman equation for household problem

$$
\begin{aligned}
V(k,z,K) = &\max_{k'}\{u(c)+\beta\sum_{z'\in Z}\pi(z'\mid z)V(k',z',G(z,K))\},\\
\text{subject to}\quad c =&(r(z,K)+1-\delta)k+w(z,K)-k',\\
k' \in&[0,(r(z,K)+1-\delta)k+w(z,K)]
\end{aligned}
$$

with the associated policy function $k' = g(k,z,K)$

Equilibrium definition

Definition 5
A **recursive competitive equilibrium** consists of household value function $V : \mathbb{R}^3_+ \to \mathbb{R}$ and policy function $g : \mathbb{R}^3_+ \to \mathbb{R}_+$, **pricing functions** $r, w : \mathbb{R}^2_+ \to \mathbb{R}_+$, and the aggregate law of motion $G : \mathbb{R}^2_+ \to \mathbb{R}_+$ such that

- (household optimization) given r, w and G, the solution to the household's Bellman equation is V with an associated policy g;
- (firm optimization) r, w satisfy $r(z, K) = zF_1(K, 1)$ and $w(z, K) = zF_2(K, 1)$, $\forall z, K$;
- (**consistency**) individual decision is consistent with aggregate law of motion

$$g(K, z, K) = G(z, K), \ \forall (z, K).$$

Straightforward to verify that market clearing conditions hold.

Equilibrium characterization

- FOC of household problem for $k' = g(k, z, K)$ (ignoring constraints)

$$u'((r(z, K) + 1 - \delta)k + w(z, K) - k') = \beta \mathbb{E}[V_1(k', z', G(z, K)) \mid z]$$

- Envelope condition at k with $k' = g(k, z, K)$

$$V_1(k, z, K) = u'((r(z, K) + 1 - \delta)k + w(z, K) - k')(r(z, K) + 1 - \delta)$$

moving one period ahead with $K' = G(z, K)$ and $k'' = g(k', z', K')$

$$V_1(k', z', K') = u'((r(z', K') + 1 - \delta)k' + w(z', K') - k'')(r(z', K') + 1 - \delta)$$

- Observe: without distinguishing k' and K', the household would perceive that their choice of k' would affect prices $r(z', k')$ and $w(z', k')$

Equilibrium characterization

- The Euler equation with $k' = g(k, z, K)$, $K' = G(z, K)$, and $k'' = g(k', z', K') = g(g(k, z, K), z', G(z, K))$

$$\begin{aligned} & u'((r(z, K) + 1 - \delta)k + w(z, K) - k') \\ =& \beta \mathbb{E}[u'((r(z', K') + 1 - \delta)k' + w(z', K') - k'')(r(z', K') + 1 - \delta) \mid z] \end{aligned}$$

- Evaluated at the equilibrium, it coincides with the planner's Euler equation
- We postpone the quantitative exercise to the next deck of slides in the context of RBC

Beyond Markov Chains (FYI)

Continuous shocks: what changes?

- Let z take values in an interval $Z \subset \mathbb{R}$, with a conditional density $\varphi(z' \mid z)$, e.g., the AR(1) for $\ln z$
- The Bellman equation replaces the sum by an integral:

$$V(k,z) = \max_{k' \in [0, zf(k)+(1-\delta)k]} \left\{ u(zf(k) + (1-\delta)k - k') + \beta \int_Z V(k', z')\, \varphi(z' \mid z)\, dz' \right\}$$

- FOC, envelope condition, functional Euler equation: unchanged, with $\sum_{z'} \pi(z' \mid z)(\cdot)$ replaced by $\int (\cdot)\, \varphi(z' \mid z)\, dz'$
- The log example goes through unchanged: $g(k,z) = \alpha\beta z k^\alpha$, and $B(z)$ is affine in $\ln z$ under the AR(1)

Why the theory still works

- We need T to map bounded continuous functions into bounded continuous functions:
  - Z bounded (e.g., a truncated AR(1)), so V is bounded
  - $\mathbb{E}[h(z') \mid z] = \int h(z')\varphi(z' \mid z)dz'$ is continuous in z for any bounded continuous h (true for the AR(1))
- Blackwell's conditions go through as before: integrals preserve monotonicity, and $\int (V + a)\varphi \, dz' = \int V\varphi \, dz' + a$. So T is a contraction: unique V, and VFI converges
- Unbounded shocks (Gaussian AR(1)): truncate, or use weighted norms. Beyond our scope
- For the curious: Stokey, Lucas, and Prescott (1989), Ch. 8 (Markov processes) and Ch. 9 (stochastic dynamic programming)

In practice: back to Markov chains

- Discretize the AR(1) into a Markov chain (Z, P) on a finite grid: Tauchen (1986), or Rouwenhorst (1995), which works better for highly persistent processes
- Then solve with VFI exactly as before. The discretization error shrinks as the grid gets finer
- So sticking to Markov chains costs little in practice

Local vs Global Solution Methods (optional)

Numerical local solution: (log-)linearization

back

- Global solutions are sometimes computationally costly
- Alternative: a "local solution," valid only when shocks are small
  1. solve for the deterministic steady state (DSS)
  2. approximate the policy functions near the DSS using the Euler equations
- (Log-)linearization: approximate policies as (log-)linear functions of states
- Especially useful when we're interested in *impulse response functions*: how endogenous variables respond to a one-time "shock" over time
- We look at *log-linearization* as an example

Numerical local solution: (log-)linearization

Look for the DSS

- Suppose $\ln z_t$ follows an AR(1) process

$$\ln z_{t+1} = (1-\rho)\mu_z + \rho \ln z_t + \epsilon_{t+1}, \quad \rho \in (0,1),\ \epsilon_{t+1} \overset{iid}{\sim} \mathcal{N}(0, (1-\rho^2)\sigma_z^2)$$

- Shut down uncertainty and let the productivity level stay at $\overline{z} := \mathbb{E}[z_t] = \exp(\mu_z + \frac{\sigma_z^2}{2})$ (not $e^{\mathbb{E}[\ln z]}$ as on the DSS/SSS slide) Euler equation at DSS same as in the deterministic case

$$1 = \beta[\overline{z} f'(k_{SS}) + 1 - \delta] \quad \Longrightarrow\ k_{SS},\ c_{SS}$$

- **Goal of log-linearization:** look for coefficients g_k and g_z such that

$$\widehat{k'} = \ln k' - \ln k_{SS} \approx g_k(\ln k - \ln k_{SS}) + g_z(\ln z - \ln \overline{z}) = g_k \widehat{k} + g_z \widehat{z}$$

Numerical local solution: (log-)linearization

- The Euler equation is

$$u'(zf(k)+(1-\delta)k-k')=\beta\mathbb{E}[u'(z'f(k')+(1-\delta)k'-k'')[z'f'(k')+1-\delta]\mid z]$$

- Total differentiation of the Euler equation wrt k, z, k', z', k'':

$$\begin{aligned}&u''(c)[f(k)dz+(zf'(k)+1-\delta)dk-dk']\\=&\beta\mathbb{E}[u''(c')[z'f'(k')+1-\delta][f(k')dz'+(z'f'(k')+1-\delta)dk'-dk'']\\&+u'(c')[f'(k')dz'+z'f''(k')dk']\mid z]\end{aligned}$$

- For any variable $x>0$, we have $dx=xd\ln x$. Evaluate EE at DSS and replace dx with $x_{SS}d\ln x$, using

$$\begin{aligned}d\ln k'=&g_k d\ln k+g_z d\ln z\\d\ln z'=&\rho d\ln z\\d\ln k''=&g_k d\ln k'+g_z d\ln z'=g_k^2 d\ln k+(g_k+\rho)g_z d\ln z\end{aligned}$$

Numerical local solution: (log-)linearization

- Let $R := \overline{z}f' + 1 - \delta$, with f, f', f'' and u'/u'' evaluated at the DSS. EE becomes

$$\begin{aligned}
&\overline{z}f\, d\ln z + Rk_{SS}\, d\ln k - k_{SS}(g_k d\ln k + g_z d\ln z) \\
=&\beta R\Big(\overline{z}f\rho\, d\ln z + Rk_{SS}(g_k d\ln k + g_z d\ln z) - k_{SS}\big(g_k^2 d\ln k + (g_k+\rho)g_z d\ln z\big)\Big) \\
&+ \beta\frac{u'}{u''}\big(\overline{z}f'\rho\, d\ln z + \overline{z}f''k_{SS}(g_k d\ln k + g_z d\ln z)\big)
\end{aligned}$$

- Gather coefficients for $d\ln k$ and $d\ln z$:

$$\begin{aligned}
R - g_k - \beta R^2 g_k + \beta R g_k^2 - \beta\frac{u'}{u''}\overline{z}f''g_k &= 0 \\
\overline{z}f - k_{SS}g_z - \beta R\big(\overline{z}f\rho + (R - g_k - \rho)k_{SS}g_z\big) - \beta\frac{u'}{u''}\big(\overline{z}f'\rho + \overline{z}f''k_{SS}g_z\big) &= 0
\end{aligned}$$

Numerical local solution: (log-)linearization

- Use the fact that $\beta R = 1$ at the DSS to simplify the expression for g_k:

$$g_k^2 - \underbrace{\left(1 + \beta^{-1} + \beta \frac{u'}{u''} \overline{z} f''\right)}_{:=B} g_k + \beta^{-1} = 0$$

  which has 2 roots, one > 1 and one $\in (0, 1)$ (note that LHS$=\beta^{-1} > 0$ at $g_k = 0$ and LHS< 0 at $g_k = 1$).

- Stationarity requires

$$g_k = \frac{B - \sqrt{B^2 - 4\beta^{-1}}}{2},$$

  and g_z follows as

$$g_z = \frac{\overline{z}}{k_{SS}} \frac{(1-\rho) f - \beta \rho u' f' / u''}{B - g_k - \rho}.$$

- So, around DSS:

$$\widehat{k'} \approx g_k \widehat{k} + g_z \widehat{z}$$

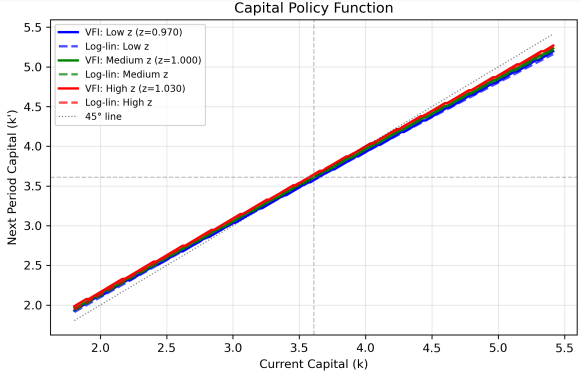
An example

- CRRA utility $u(c) = \dfrac{c^{1-\sigma} - 1}{1-\sigma}$ with $\sigma = 2$; discount factor $\beta = 0.96$
- Cobb-Douglas production function $zF(k, 1) = zk^{\alpha}$ with $\alpha = 1/3$
- Capital depreciation rate $\delta = 0.1$
- AR(1) productivity: $\ln z_{t+1} = \rho \ln z_t + (1-\rho)\mu_z + \sigma_\epsilon \epsilon_{t+1}$ with $\rho = 0.8$, $\sigma_\epsilon = 0.006$, $\epsilon_{t+1} \overset{iid}{\sim} \mathcal{N}(0,1)$, $\mu_z = -\dfrac{\sigma_z^2}{2} = -\dfrac{\sigma_\epsilon^2}{2(1-\rho^2)}$, so $\overline{z} = \mathbb{E}[z_t] = \exp(\mu_z + \frac{\sigma_z^2}{2}) = 1$.
- DSS is simply

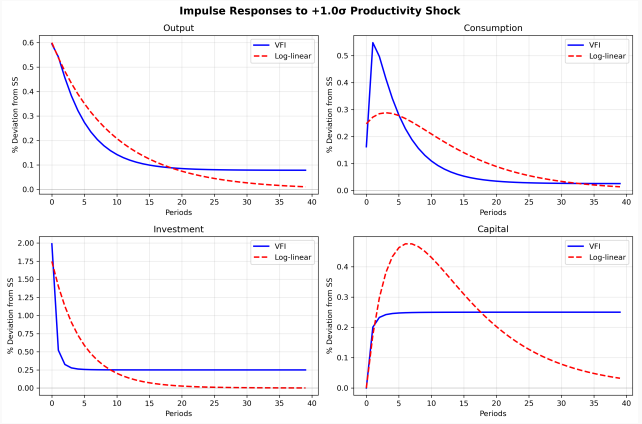
$$k_{SS} = \left(\frac{\alpha}{\beta^{-1} - (1-\delta)} \right)^{\frac{1}{1-\alpha}}, \qquad y_{SS} = k_{SS}^{\alpha}, \qquad c_{SS} = k_{SS}^{\alpha} - \delta k_{SS}, \qquad i_{SS} = \delta k_{SS}$$

An example

- Numerical global solution: VFI yields $k' = g(k, z)$
  - first, need to discretize the AR(1) process into a Markov chain
- Numerical local solution log-linearization has an analytical form: $\widehat{k'} = g_k \widehat{k} + g_z \widehat{z}$



An example: global≠good



- A 1σ shock keeps the economy near the DSS, where the policy is nearly linear: log-linearization is very accurate
- VFI with 200 grid points on $[0.5, 1.5]k_{SS}$: spacing $\approx 0.5\%$ of k_{SS}, comparable to the response of k itself. Capital gets stuck on a flat spot of the discretized policy
- Global methods pay off with large shocks, far from the DSS, or with kinks (deck 4)
- Lesson: accuracy depends on the problem and the implementation (grid, interpolation), not on "global vs local"

Readings

- AKMM Ch.7
- Krueger Ch.6
- For the curious: SLP Ch.8–9