ECON 720: Advanced Macroeconomic Theory I

Extensions and Variants of Deterministic Neoclassical Growth Model

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Fall 2026

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Overview

Goal of this set of slides:

1. to see how the basic neoclassical growth model can be altered/enriched
2. to get comfortable with dynamic programming and recursive competitive equilibria

Plan and readings:

- consumption-saving decisions (AKMM Ch.4.3, Ch.11.3.1)
- occasionally binding constraints: irreversible investment, borrowing limits
- heterogeneous households: a simple case
- endogenous labor supply (AKMM Ch.5.4.2, Ch.12.3) (optional)
- fiscal policy (AKMM Ch.15.1-15.4) (optional)
- human capital (optional)

Consumption-Saving Decisions

Setup

We take a small deviation from neoclassical growth and focus on household consumption-saving/borrowing decisions for a moment

- $t = 0, 1, 2, \ldots$
- Agent: one household with lifetime utility

$$U(\{c_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

  where $\beta \in (0, 1)$, $u' > 0$, $u'' < 0$, Inada conditions: $u'(0) = \infty$, $u'(\infty) = 0$

- $\forall t$, household receives an endowment (or wage) $w_t \geq 0$ of period-t consumption good
- Household has perfect foresight

Setup

- $\forall t$, household has access to **borrowing and lending** at net interest rate $r_t = r$
- Household decides on their asset position a_{t+1} one period ahead, $\forall t$, with a_0 given:

$$\begin{aligned} a_t > 0: &\quad \text{lending or saving} \\ a_t < 0: &\quad \text{borrowing or in debt} \end{aligned}$$

- Household budget constraint at t is

$$c_t + a_{t+1} \leq w_t + (1+r)a_t$$

assume $\sum_{t=0}^{\infty} \frac{w_t}{(1+r)^t} < \infty$, $a_0 > -\sum_{t=0}^{\infty} \frac{w_t}{(1+r)^{t+1}}$

- Without loss, let $a_0 = 0$
- We focus on the household's decision

Household problem

- We impose a **no Ponzi game** (nPg) or no Ponzi scheme condition:

$$\lim_{t\to\infty} \frac{a_{t+1}}{(1+r)^t} \geq 0$$

  so household cannot roll over debts indefinitely to achieve ∞ utility

- Counterpart in finite horizon case $T < \infty$: household can't die in debt

$$a_{T+1} \geq 0$$

  without it, household would want to set $a_{T+1} = -\infty$!

- This is a **constraint**, not the same as TVC, which is an optimization condition
  - nPg constraint: "can't borrow in the end"
  - TVC: "don't save in the end"

Household problem

- Household problem

$$\begin{aligned} \max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{subject to} \quad c_t + a_{t+1} \leq & w_t + (1+r)a_t, \ \forall t, \ \text{given } a_0 = 0, \\ c_t \geq & 0, \ \forall t \\ \lim_{t\to\infty} \frac{a_{t+1}}{(1+r)^t} \geq & 0. \end{aligned}$$

- Can't attach any multiplier to the nPg constraint. Good news: it never binds when the maximization problem is well-defined (so the usual FOC analysis applies).
- nPg simply ensures a well-defined maximization problem

nPg constraint

nPg constraint implies a well-defined **lifetime budget constraint**

- Rewrite period-t constraint as

$$\frac{c_t}{(1+r)^t} + \frac{a_{t+1}}{(1+r)^t} \leq \frac{w_t}{(1+r)^t} + \frac{a_t}{(1+r)^{t-1}}$$

- Sum up from $t = 0$ to any T

$$\sum_{t=0}^{T} \frac{c_t}{(1+r)^t} + \frac{a_{T+1}}{(1+r)^T} \leq \sum_{t=0}^{T} \frac{w_t}{(1+r)^t}$$

- Sending $T \to \infty$ yields a lifetime budget constraint

$$\sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} \leq \sum_{t=0}^{\infty} \frac{w_t}{(1+r)^t} - \lim_{T\to\infty} \frac{a_{T+1}}{(1+r)^T} \leq \sum_{t=0}^{\infty} \frac{w_t}{(1+r)^t}$$

Natural borrowing limit

- Imposing a **natural borrowing limit** is equivalent to imposing the **nPg constraint**

$$a_{t+1} \geq -\sum_{s=1}^{\infty} \frac{w_{t+s}}{(1+r)^s}$$

  household cannot borrow more than the present value of all future income

- See AKMM Ch.4.A.1 for a proof.
- Natural borrowing limit never binds but ensures a well-defined maximization problem

Analysis

- Euler equations

$$u'(c_t) = \beta u'(c_{t+1})(1+r), \quad \forall t$$

  which immediately implies

$$\begin{aligned} c_{t+1} &> c_t \quad \text{if } \beta(1+r) > 1, \\ c_{t+1} &= c_t \quad \text{if } \beta(1+r) = 1, \\ c_{t+1} &< c_t \quad \text{if } \beta(1+r) < 1. \end{aligned}$$

- The growth of c_t does not depend on the timing of income w_t: **consumption smoothing**. Why?

A two-period tradeoff inside the infinite horizon

- Keep all other periods at the optimum: a_t^*, a_{t+2}^*, and c_s for $s \neq t, t+1$. Then (c_t^*, c_{t+1}^*) must solve the *two-period* problem

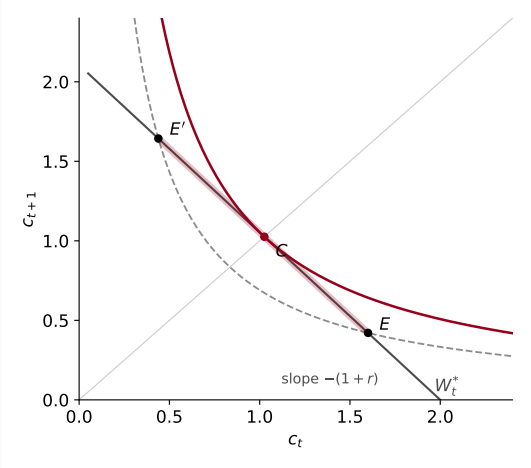
$$\begin{aligned} &\max_{c_t, c_{t+1}, a_{t+1}} \{u(c_t) + \beta u(c_{t+1})\} \\ \text{s.t.} \quad & c_t + a_{t+1} = w_t + (1+r)a_t^* \\ & c_{t+1} + a_{t+2}^* = w_{t+1} + (1+r)a_{t+1} \end{aligned}$$

- Eliminate a_{t+1}: the two constraints become one,

$$c_t + \frac{c_{t+1}}{1+r} = W_t^* := w_t + (1+r)a_t^* + \frac{w_{t+1} - a_{t+2}^*}{1+r}$$

- Tangency, MRS = relative price, is the Euler equation: $\dfrac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + r$

Why smooth? Convex preferences



- $u(c_t) + \beta u(c_{t+1})$ is strictly quasiconcave: upper contour sets are strictly convex
- E: $a_{t+1} = 0$, income high at t, low at $t+1$. E' is on the same indifference curve and the same budget line
- Every point strictly between E and E' is affordable and strictly better: averages beat extremes. So the optimum C is smoother
- Quasiconcavity is *ordinal*: smoothing comes from convex preferences. The curvature of indifference curves (EIS $= 1/\sigma$ for CRRA) governs how much

CRRA

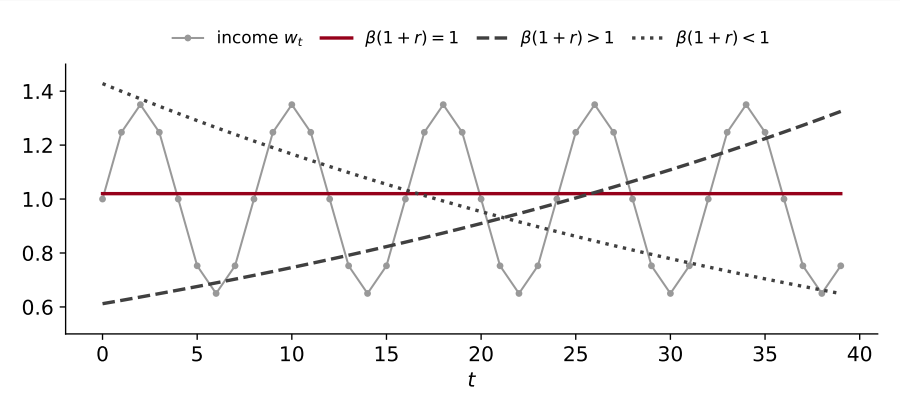
Smoothing without FOCs: Jensen's inequality

- Let $\beta(1+r)=1$, so prices are $(1+r)^{-t}=\beta^t$, and let $\bar{c}:=(1-\beta)\sum_t \beta^t w_t$, the constant consumption the household can afford
- For any feasible $\{c_t\}$, i.e., $\sum_t \beta^t c_t \le \sum_t \beta^t w_t$, since $(1-\beta)\beta^t$ are weights summing to 1,

$$\sum_{t=0}^{\infty}\beta^t u(c_t)=\frac{1}{1-\beta}\sum_{t=0}^{\infty}(1-\beta)\beta^t u(c_t)\le\frac{1}{1-\beta}\,u\Big((1-\beta)\sum_{t=0}^{\infty}\beta^t c_t\Big)\le\frac{u(\bar{c})}{1-\beta}$$

  strictly if u is strictly concave and $\{c_t\}$ is not constant. The flat path is optimal
- Caution: this uses *time separability* and a *concave* u, the same in every period
- Replace the weights $(1-\beta)\beta^t$ by probabilities: $\mathbb{E}[u(c)]\le u(\mathbb{E}[c])$ is risk aversion. With time-separable expected utility, one u governs both: EIS $=1/\sigma$ and relative risk aversion $=\sigma$. More when we add uncertainty CRRA

Tilt vs smoothing



- Curvature smooths out the income fluctuations; $\beta(1+r)$ tilts the path
- Same income in all three cases: only its present value matters

Transversality condition?

- Transversality condition for this model is

$$\lim_{t\to\infty} \beta^t u'(c_t)(1+r)a_t \leq 0$$

- Repeatedly applying the Euler equations leads to

$$\lim_{t\to\infty} \beta^t u'(c_t)(1+r)a_t = \lim_{t\to\infty} \beta^{t-1} u'(c_{t-1})a_t = \ldots = u'(c_0) \lim_{t\to\infty} \frac{a_t}{(1+r)^{t-1}} \leq 0$$

  so TVC can be re-written as

$$\lim_{t\to\infty} \frac{a_t}{(1+r)^{t-1}} \leq 0$$

- nPg constraint with inequality in the *opposite* direction!

The corresponding Arrow-Debreu market structure

- Markets for consumption goods in all periods open at $t = 0$, with prices $\{p_t\}_{t=0}^{\infty}$
- Household problem becomes

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

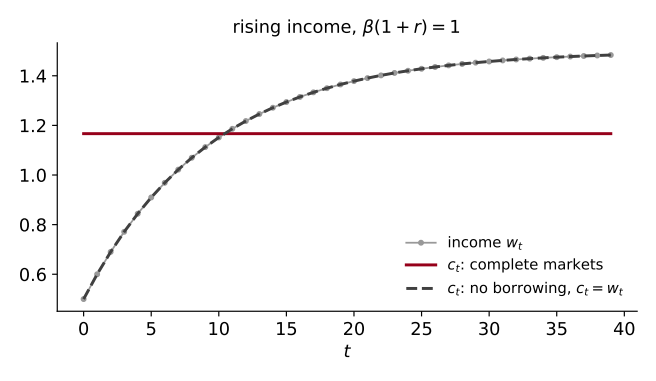
$$\text{subject to} \quad \sum_{t=0}^{\infty} p_t c_t \leq \sum_{t=0}^{\infty} p_t w_t,$$

$$c_t \geq 0, \ \forall t$$

- Easy to see that if $p_t = \dfrac{1}{(1+r)^t}$, the household will choose the same allocation in this AD scenario and the sequential markets setting with borrowing/saving

Market completeness

- With nPg, one bond per period gives the same budget set as AD: markets are **complete**. Only the PV of income matters
- Incomplete markets, e.g., no borrowing $a_{t+1} \geq 0$ (PS3 Q1(c)): $u'(c_t) \geq \beta(1+r)u'(c_{t+1})$, "=" if $a_{t+1} > 0$



An example

- CRRA utility: $u(c) = \dfrac{c^{1-\sigma} - 1}{1 - \sigma}$
- Euler equations imply

$$c_{t+1} = [\beta(1+r)]^{1/\sigma} c_t = [\beta(1+r)]^{(t+1)/\sigma} c_0$$

- Lifetime budget constraint then implies (assume $\beta^{1/\sigma}(1+r)^{1/\sigma - 1} < 1$)

$$c_0 \sum_{t=0}^{\infty} \frac{[\beta(1+r)]^{t/\sigma}}{(1+r)^t} = \sum_{t=0}^{\infty} \frac{w_t}{(1+r)^t}$$

$$\implies c_0 = \underbrace{\left[1 - \beta^{1/\sigma}(1+r)^{1/\sigma - 1}\right]}_{\text{marginal propensity to consume (MPC)}} \times \underbrace{\left(\sum_{t=0}^{\infty} \frac{w_t}{(1+r)^t}\right)}_{\text{PV of wealth}}$$

permanent income hypothesis (deterministic case)

The MPC and permanent income

- MPC out of wealth: $1-\beta^{1/\sigma}(1+r)^{1/\sigma-1}$. If $\beta(1+r)=1$: $\dfrac{r}{1+r}$ for any σ (about 0.04 if $r=4\%$)
- With $\beta(1+r)=1$: a one-time windfall Δ at $t=0$ raises every c_t by $\frac{r}{1+r}\Delta$; a permanent rise of Δ in every w_t raises every c_t by Δ
- A higher r: income vs substitution effects again. $\sigma=1$: MPC $=1-\beta$; $\sigma>1$: MPC rises with r (income effect dominates); $\sigma<1$: MPC falls. Plus a wealth effect: the PV of future income falls
- Data: households spent 20–40% of the 2001 tax rebates on nondurables within a quarter (Johnson, Parker and Souleles, 2006), far above $\frac{r}{1+r}$. This points to borrowing constraints and incomplete markets (next semester)

Occasionally Binding Constraints

What if the Bellman equation has a binding constraint?

$$V(s) = \max_a \left\{ F(s,a) + \beta V(\varphi(s,a)) \right\} \quad \text{s.t. } h(s,a) \geq 0$$

- Taking V as given, the RHS is a *static* problem in a. If it is concave, KKT is necessary and sufficient:

$$\mathcal{L} = F(s,a) + \beta V(\varphi(s,a)) + \mu\, h(s,a)$$

- FOC: $F_a + \beta V'(\varphi)\varphi_a + \mu h_a = 0$, with $\mu \geq 0$, $\mu h = 0$
- EC: $V'(s) = F_s + \beta V'(\varphi)\varphi_s + \mu h_s$
- So μ enters the envelope condition if and only if the constraint depends on the state
- Two illustrations: irreversible investment (technology, planner) and a borrowing limit (markets, household)

Illustration 1: irreversible investment

Back to the NGM, with one change: investment cannot be negative.

- Impose $i_t \geq 0$. Since $k_{t+1} = (1-\delta)k_t + i_t$, this means

$$k_{t+1} \geq (1-\delta)k_t \quad \Longrightarrow \quad \Gamma(k) = \left[(1-\delta)k,\ \widetilde{f}(k)\right], \quad \widetilde{f}(k) := f(k) + (1-\delta)k$$

- The lower bound moves with the state: the constraint binds *occasionally*
- If $\delta = 1$, the constraint is $k' \geq 0$ and never binds

Applying the recipe

$$V(k) = \max_{k'} \left\{ u(\widetilde{f}(k) - k') + \beta V(k') \right\} \quad \text{s.t. } h(k, k') = k' - (1-\delta)k \geq 0$$

- FOC: $u'(c) = \beta V'(k') + \mu$, with $\mu \geq 0$, $\mu\left[k' - (1-\delta)k\right] = 0$
- EC: $h_k = -(1-\delta) \neq 0$, so $V'(k) = u'(c)\,\widetilde{f}'(k) - \mu(1-\delta)$
- Euler equation:
$$u'(c) = \beta\left[u'(c')\,\widetilde{f}'(k') - \mu'(1-\delta)\right] + \mu$$
- Same as from the sequential problem, attaching $\beta^t \mu_t$ to $k_{t+1} - (1-\delta)k_t \geq 0$

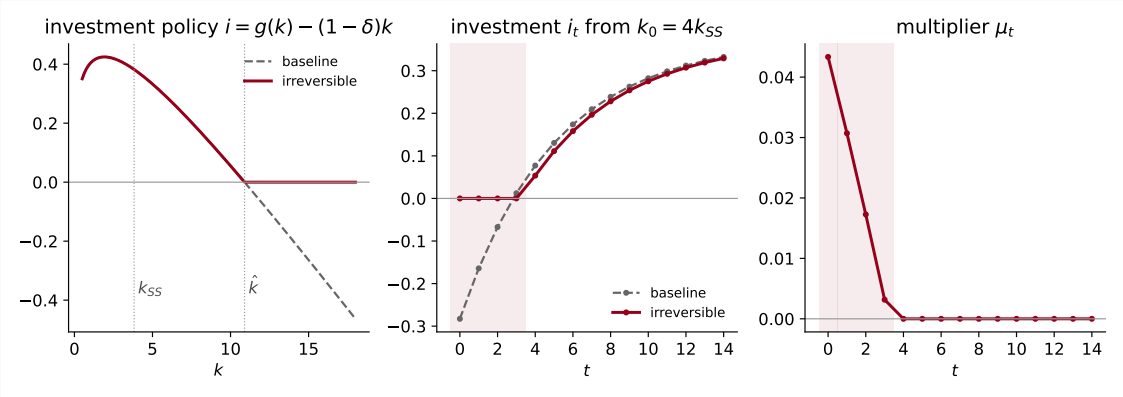
Why the cookbook still works

- V is concave: u, $\widetilde{f}$ are concave and the *graph* of Γ is convex
- Differentiability: Benveniste–Scheinkman needs k' in the *interior* of $\Gamma(k)$, which fails when $\mu > 0$. V is still differentiable here (Rincón-Zapatero and Santos, 2009)
- What does break the cookbook: non-convexities (fixed costs, indivisibilities). V can have kinks, and FOCs are no longer sufficient (Milgrom and Segal, 2002)

When does it bind?

- Far above k_{SS}, the planner wants to run capital down fast, but depreciation alone lowers k by only a share δ per period. It binds exactly where the baseline policy would set $i < 0$
- $\mu > 0$: $i = 0$, so $k' = (1-\delta)k$ and $c = \widetilde{f}(k) - k' = f(k)$, and $u'(c) > \beta V'(k')$. The planner wants to disinvest but cannot
- Example (computed): $u = \ln c$, $f(k) = k^{0.36}$, $\beta = 0.95$, $\delta = 0.1$
  - $k_{SS} = 3.82$, $i_{SS} = 0.38 > 0$: slack at the steady state
  - binds if and only if $k > \hat{k} = 10.9 \approx 2.85\, k_{SS}$
- Preview: irreversibility gets a lot more interesting with uncertainty: an option value of waiting (Bernanke, 1983; Dixit and Pindyck, 1994)

Irreversibility in action



- $k \leq \hat{k}$: the baseline path never has $i < 0$, so it is still optimal
- From $k_0 = 4k_{SS}$: $i_t = 0$, $\mu_t > 0$ for $t \leq 3$ (shaded), then baseline dynamics
- Same example in continuous time later

Computing with occasionally binding constraints

- VFI: just restrict the choice grid to $\Gamma(k)$. Nothing else changes
- Euler-equation methods: must also find the binding region. Here: solve the baseline, then $\hat{k}$ is where its investment hits 0
- Linearization around the steady state misses the kink entirely, since the constraint is slack there
- One reason global methods (VFI, policy function iteration, endogenous grid method) dominate in modern macro

Interpretation: Tobin's q

- Let $q := \dfrac{\beta V'(k')}{u'(c)}$: the value of one more unit of installed capital, in units of today's consumption. The FOC becomes

$$q = 1 - \frac{\mu}{u'(c)} \leq 1$$

  - $i > 0$: $q = 1$. Invest until installed capital is worth its replacement cost
  - $i = 0$: $q < 1$. Installed capital is worth less than new capital: nobody invests
- In a competitive equilibrium (e.g., firms own capital, as in PS3 Q3), q is the market price of installed capital
- Add an investment cap $i \leq \bar{\imath}$ (multiplier ν): $q = 1 + \frac{\nu - \mu}{u'(c)}$, so $q > 1$ when the cap binds. Installed capital is worth more than new capital, but no more can be added
- Without such constraints, $q \equiv 1$. Capital adjustment costs make q move smoothly: continuous time

Illustration 2: a borrowing limit

- The consumption-saving household (constant w) with no borrowing:

$$W(a) = \max_{a'} \left\{ u(w + (1+r)a - a') + \beta W(a') \right\} \quad \text{s.t. } h(a, a') = a' \geq 0$$

- $h_a = 0$: the EC is the usual $W'(a) = (1+r)u'(c)$, and $u'(c) = \beta(1+r)u'(c') + \mu$
- Technology vs markets
  - irreversibility restricts the *production set*: the constrained planner allocation is efficient and can be decentralized
  - a borrowing limit restricts *trades*: markets are incomplete
- With a representative household, $a' \geq 0$ changes prices, not allocations (PS3 Q1(c)). It bites with heterogeneity: next semester

Heterogeneous Households

Two types of households

- Consider the simple neoclassical growth model but with two types of households $i = A, B$
- The measure of type-i households is $\phi_i > 0$, $i = A, B$, with $\phi_A + \phi_B = 1$
- All households are identical at 0 except for type-specific initial capital stock $k_0^A \neq k_0^B > 0$
- How to write household problem in recursive formulation?

Household problem in recursive formulation

For each household type $i = A, B$

- State variables: k, K^A, K^B
- Control variable: k'
- Pricing functions: $r(K)$, $w(K)$
- Aggregate laws of motion: $K^{A'} = G^A(K^A, K^B)$, $K^{B'} = G^B(K^A, K^B)$
- Household Bellman equation

$$V(k, K^A, K^B) = \max_{k' \geq 0} \{u(c) + \beta V(k', G^A(K^A, K^B), G^B(K^A, K^B))\}$$

$$\text{where } c = [r(K) + 1 - \delta]k + w(K) - k' \geq 0, \quad K = \phi_A K^A + \phi_B K^B$$

with associated policy $k' = g(k, K^A, K^B)$

Recursive competitive equilibrium

Definition 1

A **recursive competitive equilibrium** consists of household value function $V : \mathbb{R}^3_+ \to \mathbb{R}$ and policy function $g : \mathbb{R}^3_+ \to \mathbb{R}_+$, **pricing functions** $r, w : \mathbb{R}_+ \to \mathbb{R}_+$, and the aggregate laws of motion $G^A, G^B : \mathbb{R}^2_+ \to \mathbb{R}_+$ such that

- (household optimization) given r, w and G^A, G^B, the solution to the household's Bellman equation is V with associated policy g;
- (firm optimization) r, w satisfy $r(K) = F_1(K, 1)$ and $w(K) = F_2(K, 1)$, $\forall K$;
- (**consistency**) individual decision is consistent with aggregate law of motion

$$g(K^A, K^A, K^B) = G^A(K^A, K^B), \quad g(K^B, K^A, K^B) = G^B(K^A, K^B), \quad \forall K^A, K^B.$$

Straightforward to verify that market clearing conditions hold.

What is the aggregate state?

- Prices depend only on K, but $K' = \phi_A\, g(K^A, \cdot) + \phi_B\, g(K^B, \cdot)$ depends on how K is split. To forecast future prices, households need (K^A, K^B)
- (K^A, K^B) *is* the cross-sectional distribution of capital: mass ϕ_A at K^A and ϕ_B at K^B
- N types: the state is $(K^1, \ldots, K^N)$. A continuum of households: the state is a distribution Φ, an infinite-dimensional object (next semester: Aiyagari, Krusell–Smith)
- Special case: if g is affine in k with a common slope (Gorman aggregation), K alone is enough, e.g., CRRA u (next slide). Not in general

FYI: a special case where K is enough

CRRA u, same β, σ for all, and $k' \geq 0$ never binds. Let $R_t := r_t + 1 - \delta$

- Euler: $c_{i,t+1} = (\beta R_{t+1})^{1/\sigma} c_{it}$. The growth rate is common to all i
- Lifetime budget from t on, with H_t the PV of wages from t on (common):

$$\sum_{s \geq t} p_{s|t}\, c_{is} = R_t k_{it} + H_t, \qquad p_{s|t} := \prod_{\tau=t+1}^{s} R_\tau^{-1}$$

  so $c_{it} = m_t \left(R_t k_{it} + H_t\right)$ with a common MPC m_t (log: $m_t = 1 - \beta$)
- Policy: $k_{i,t+1} = R_t k_{it} + w_t - c_{it} = (1 - m_t) R_t\, k_{it} + w_t - m_t H_t$: affine in k_{it}, common slope
- Aggregate with weights ϕ_i:

$$K_{t+1} = (1 - m_t) R_t\, K_t + w_t - m_t H_t$$

  Same as the representative household: the distribution does not affect aggregates (Chatterjee, 1994). It still evolves: $k_{i,t+1} - K_{t+1} = (1 - m_t) R_t (k_{it} - K_t)$

Endogenous Labor Supply (optional)

Introducing labor disutility

Back to a representative household

- So far, labor supply is assumed to be inelastic/exogenous
- How to model endogenous labor supply of households?
- Let the household value both consumption and leisure, with instantaneous utility

$$u(c, 1-l) \overset{e.g.}{=} u(c) + \widetilde{v}(1-l)$$

where c is consumption, l is hours worked, and total time endowed is normalized to 1.

- It's also common to assume a separable labor disutility term

$$u(c) - v(l) \overset{e.g.}{=} \ln c - \psi \frac{l^{1+1/\gamma}}{1+1/\gamma}, \gamma > 0$$

We'll use this form and assume $v' > 0$, $v'' > 0$.

Social planner's problem in sequential formulation

Planner's problem at $t = 0$ can be written as (ignoring non-negativity constraints)

$$\begin{aligned} W(\overline{k}_0) = & \max_{\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t [u(c_t) - v(l_t)] \\ \text{subject to} \quad & c_t + k_{t+1} \leq F(k_t, l_t) + (1-\delta)k_t, \forall t \end{aligned}$$

The Lagrangian can be written as (ignoring non-negativity constraints)

$$\mathcal{L} = \sum_{t=0}^{\infty} \left(\beta^t [u(c_t) - v(l_t)] + \lambda_t [F(k_t, l_t) + (1-\delta)k_t - c_t - k_{t+1}] \right)$$

Optimization conditions

- Straightforward to show that the optimization conditions include the usual Euler equations for *inter*-temporal considerations

$$u'(c_t) = \beta u'(c_{t+1})[F_1(k_{t+1}, l_{t+1}) + 1 - \delta], \ \forall t$$

- and those for *intra*-temporal considerations

$$\underbrace{v'(l_t)}_{\substack{\text{marginal disutility if} \\ \text{working 1 more hour at } t}} = u'(c_t) \times \underbrace{F_2(k_t, l_t)}_{\text{MPL at } t} \iff MRS(l_t, c_t) = \frac{v'(l_t)}{u'(c_t)} = F_2(k_t, l_t), \ \forall t$$

- TVC has the same form $\lim_{t\to\infty} \lambda_t k_{t+1} = \lim_{t\to\infty} \beta^t u'(c_t) k_{t+1} = 0$

Social planner's problem in recursive formulation

- State variable: k
- Control variables: l, k'
- Bellman equation

$$V(k) = \max_{l,k' \geq 0} \{u(c) - v(l) + \beta V(k')\} \quad \text{where } c = F(k,l) + (1-\delta)k - k' \geq 0$$

with associated policies $k' = g^K(k)$ and $l = g^L(k)$

- Optimization conditions at $l = g^L(k),\ k' = g^K(k),\ l' = g^L(g^K(k)),\ k'' = g^K(g^K(k))$

$$\begin{aligned} v'(l) =& u'(F(k,l) + (1-\delta)k - k')F_2(k,l) \\ u'(F(k,l) + (1-\delta)k - k') =& \beta V'(k') = \beta u'(F(k',l') + (1-\delta)k' - k'')[F_1(k',l') + 1 - \delta] \end{aligned}$$

same as in the sequential formulation

Steady state

- A **steady state** is fully characterized by a $k_{SS} > 0$ such that $g^K(k_{SS}) = k_{SS}$.
- At the steady state, we still have the modified golden rule, except for the labor input

$$1 = \beta[F_1(k_{SS}, l_{SS}) + 1 - \delta]$$

  where $l_{SS} = g^L(k_{SS})$ and k_{SS} must also satisfy

$$v'(l_{SS}) = u'(F(k_{SS}, l_{SS}) - \delta k_{SS}) F_2(k_{SS}, l_{SS})$$

- Special case: $u(c) = \ln c$, $v(l) = \psi \frac{l^{1+1/\gamma}}{1+1/\gamma}$, $\delta = 1$, $F(k,l) = k^\alpha l^{1-\alpha}$: $k_{SS}/l_{SS} = (\alpha\beta)^{\frac{1}{1-\alpha}}$, $l_{SS} = \left(\frac{1-\alpha}{\psi(1-\alpha\beta)}\right)^{\frac{\gamma}{1+\gamma}}$

Sequential markets

- Consider sequential markets with perfect competition
- Firm problem remains unchanged
- Household problem becomes (ignoring non-negativity constraints and zero profits)

$$\max_{\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t [u(c_t) - v(l_t)]$$

$$\text{subject to} \quad c_t + k_{t+1} \leq (r_t + 1 - \delta) k_t + w_t l_t, \ \forall t$$

Sequential markets equilibrium

Definition 2

A **sequential markets equilibrium** consists of a sequence of prices $\{r_t^*, w_t^*\}_{t=0}^{\infty}$, household allocation $\{c_t^*, l_t^*, k_{t+1}^*\}_{t=0}^{\infty}$, and firm allocation $\{K_t^d, L_t^d\}_{t=0}^{\infty}$, such that

- (Household optimization) given $\{r_t^*, w_t^*\}_{t=0}^{\infty}$, $\{c_t^*, l_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ solves the household's problem;
- (Firm optimization) given $\{r_t^*, w_t^*\}_{t=0}^{\infty}$, $\{K_t^d, L_t^d\}_{t=0}^{\infty}$ solves the firm's per-period problem $\forall t$;
- (Market clearing) all markets clear in every period t:

$$c_t^* + k_{t+1}^* = F(K_t^d, L_t^d) + (1-\delta)k_t^*, \qquad k_t^* = K_t^d, \qquad l_t^* = L_t^d, \qquad \forall t.$$

Analysis

- Household FOCs

$$\begin{aligned}\beta^t u'(c_t) &= \lambda_t \\ \beta^t v'(l_t) &= \lambda_t w_t \\ \lambda_t &= \lambda_{t+1}(r_{t+1} + 1 - \delta)\end{aligned}$$

  which gives us the Euler equations

$$u'(c_t) = \beta u'(c_{t+1})(r_{t+1} + 1 - \delta)$$

  and the intratemporal conditions for labor supply

$$v'(l_t) = u'(c_t)w_t \quad \iff \quad MRS(l_t, c_t) = w_t$$

- And the transversality condition: $\lim_{t\to\infty} \left(\prod_{s=0}^{t-1} \frac{1}{r_{s+1}+1-\delta}\right) k_{t+1} = 0$

Equilibrium prices

- Firm optimization at t given r_t, w_t implies

$$r_t = F_1(K_t^d, L_t^d), \quad w_t = F_2(K_t^d, L_t^d)$$

- Evaluated at the SME

$$r_t^* = F_1(k_t^*, l_t^*), \quad w_t^* = F_2(k_t^*, l_t^*)$$

Recursive formulation of household problem

- State variables: k, K
- Pricing equations: $r(K), w(K)$
- Aggregate labor: $L(K)$
- Aggregate law of motion: $K' = G(K)$
- Given r, w, G, household Bellman equation can be written as

$$V(k, K) = \max_{l, k' \geq 0} \{u(c) - v(l) + \beta V(k', G(K))\}$$
$$\text{where } c = [r(K) + 1 - \delta]k + w(K)l - k' \geq 0$$

with associated policy functions $l = g^L(k, K)$ and $k' = g^K(k, K)$

Recursive competitive equilibrium

Definition 3

A **recursive competitive equilibrium** consists of household value function $V : \mathbb{R}_+^2 \to \mathbb{R}$ and policy functions $g^K, g^L : \mathbb{R}_+^2 \to \mathbb{R}_+$, **pricing functions** $r, w : \mathbb{R}_+ \to \mathbb{R}_+$, aggregate labor $L : \mathbb{R}_+ \to [0, 1]$, and the aggregate law of motion $G : \mathbb{R}_+ \to \mathbb{R}_+$ such that

- (household optimization) given r, w and G, the solution to the household's Bellman equation is V with associated policies g^K, g^L;
- (firm optimization) r, w satisfy $r(K) = F_1(K, L(K))$ and $w(K) = F_2(K, L(K))$, $\forall K$;
- (**consistency**) individual decision is consistent with aggregate law of motion

$$g^K(K, K) = G(K), \quad g^L(K, K) = L(K), \quad \forall K.$$

Straightforward to verify that market clearing conditions hold.

Income and Substitution Effects in Labor Supply (optional)

How does labor supply respond to wages?

- A particular question with elastic labor supply:

  How does labor supply respond to changes in market wages?

- An illustrative static labor-leisure choice...

$$\max_{c,l\geq 0}\{u(c)-v(l)\} \quad \text{s.t. } c\leq wl+m$$

  where m is non-labor income.

- FOC immediately implies

$$v'(l)=u'(c)w=u'(wl+m)w$$

  which implicitly defines a **labor supply** function: $l(w,m)$. So: $\dfrac{\partial l}{\partial w} >,=,<0$?

How does labor supply respond to wages?

- Two opposing effects of an increase in w on l:
  - **substitution effect (+)**: incentive to work more b/c greater marginal reward to working
  - **income effect (−)**: incentive to work less b/c marginal utility of more consumption is lower with greater income
- With separable consumption utility and labor disutility, it is straightforward to show (we can use Slutsky equation too)

$$\frac{\partial l(w,m)}{\partial w} = \underbrace{\frac{u'}{v'' - u''w^2}}_{\substack{>0, \\ \text{substitution effect}}} + \underbrace{\frac{u''wl}{v'' - u''w^2}}_{\substack{=\frac{\partial l(w,m)}{\partial m} \times l < 0, \\ \text{income effect}}}$$

  and the sign depends on which effect dominates

Government and Taxes (optional)

Introducing a government

What does a government do?

- **Government spending/expenditure** in period t: G_t
  - Starting point: G_t is useless and "thrown into the ocean", and it's exogenous
  - Alternative (we won't discuss it here): G_t enters household utility (public services) and/or firm production technology (infrastructure)
- How is G_t funded? (Fiscal policies)
  - Tax revenue
  - Government bond issuance

We add a government to the neoclassical growth framework.

Taxes only

- Ignore government bonds for now
- Consider an *exogenous* government policy sequence

$$\{\tau_{ct}, \tau_{kt}, \tau_{lt}, G_t, T_t\}_{t=0}^{\infty}$$

where

  - $\tau_{ct} \geq 0$: tax rate on household consumption expenditure
  - $\tau_{kt}, \tau_{lt} \in [0, 1]$: (net) capital income tax rate and labor income tax rate
  - $G_t \geq 0$: government expenditure in period-t consumption good
  - T_t: if $T_t > 0$: government transfer to household; if $T_t < 0$: lump sum tax

Sequential markets

- Resource constraint/consumption good market clearing at t

$$C_t + G_t + K_{t+1} = F(K_t, L_t) + (1-\delta)K_t$$

- Firm problem does not change: $\max_{K,L>0}\{F(K,L) - r_t K - w_t L\}$, $\forall t$

- Suppose household *supplies labor endogenously*. Household problem becomes

$$\max_{\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t [u(c_t) - v(l_t)]$$

$$\text{subject to} \quad (1+\tau_{ct})c_t + k_{t+1} \leq (1-\tau_{lt})w_t l_t + k_t + (1-\tau_{kt})(r_t - \delta)k_t + T_t, \ \forall t$$

- Government does not make any decisions but is subject to a per-period constraint of balanced budget

$$G_t + T_t = \tau_{ct} C_t + \tau_{kt}(r_t - \delta)K_t + \tau_{lt} w_t L_t$$

Sequential markets equilibrium

Definition 4
Given a government policy sequence $\{\tau_{ct}, \tau_{kt}, \tau_{lt}, G_t, T_t\}_{t=0}^{\infty}$, a **sequential markets equilibrium** consists of a sequence of prices $\{r_t^*, w_t^*\}_{t=0}^{\infty}$, household allocation $\{c_t^*, l_t^*, k_{t+1}^*\}_{t=0}^{\infty}$, and firm allocation $\{K_t^d, L_t^d\}_{t=0}^{\infty}$, such that

- (Household optimization) given $\{r_t^*, w_t^*\}_{t=0}^{\infty}$, household optimizes at $\{c_t^*, l_t^*, k_{t+1}^*\}_{t=0}^{\infty}$;
- (Firm optimization) given $\{r_t^*, w_t^*\}_{t=0}^{\infty}$, firm optimizes at $\{K_t^d, L_t^d\}_{t=0}^{\infty}$, $\forall t$;
- (Government balanced budget) government carries no deficit or surplus;
- (Market clearing) all markets clear in every period t:

$$c_t^* + k_{t+1}^* + G_t = F(K_t^d, L_t^d) + (1-\delta)k_t^*, \qquad k_t^* = K_t^d, \qquad l_t^* = L_t^d, \qquad \forall t.$$

Note that gov balanced budget condition is redundant when household optimizes and good market clears.

Equilibrium characterization

- Let λ_t be the Lagrangian multiplier on budget constraint at t. Household FOCs become

$$\begin{aligned}\beta^t u'(c_t) =& \lambda_t(1+\tau_{ct})\\ \beta^t v'(l_t) =& \lambda_t(1-\tau_{lt})w_t\\ \lambda_t =& \lambda_{t+1}[1+(1-\tau_{k,t+1})(r_{t+1}-\delta)]\end{aligned}$$

  together with a TVC $\lim_{t\to\infty}\lambda_t k_{t+1}=0$.

- FOCs imply:

$$\begin{aligned}\frac{v'(l_t^*)}{u'(c_t^*)} =& \frac{1-\tau_{lt}}{1+\tau_{ct}}w_t^* \overset{eqm}{=} \frac{1-\tau_{lt}}{1+\tau_{ct}}F_2(k_t^*,l_t^*)\\ \frac{u'(c_t^*)}{\beta u'(c_{t+1}^*)} =& \frac{1+\tau_{ct}}{1+\tau_{c,t+1}}[1+(1-\tau_{k,t+1})(r_{t+1}^*-\delta)] \overset{eqm}{=} \frac{1+\tau_{ct}}{1+\tau_{c,t+1}}[1+(1-\tau_{k,t+1})(F_1(k_{t+1}^*,l_{t+1}^*)\end{aligned}$$

  Is the equilibrium outcome efficient?

Planner's problem

- If a social planner also *takes as given* $\{G_t\}_{t=0}^{\infty}$:

$$\begin{aligned} &\max_{\{c_t,l_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t [u(c_t) - v(l_t)] \\ \text{subject to} \quad & c_t + G_t + k_{t+1} \leq F(k_t, l_t) + (1-\delta)k_t, \ \forall t \end{aligned}$$

- So a Pareto optimal allocation must satisfy the planner's FOCs:

$$\begin{aligned} \frac{v'(l_t^{**})}{u'(c_t^{**})} &= F_2(k_t^{**}, l_t^{**}) \\ \frac{u'(c_t^{**})}{\beta u'(c_{t+1}^{**})} &= [1 + F_1(k_{t+1}^{**}, l_{t+1}^{**}) - \delta] \end{aligned}$$

The transversality condition is $\lim_{t\to\infty} \beta^t u'(c_t) k_{t+1} = 0$.

Proportional taxes are distortionary

- Household's and planner's optimal conditions are identical only if

$$\tau_{ct} = \tau_{lt} = \tau_{kt} = 0, \ \forall t$$

(recall τ's are taxes, ≥ 0; this would change if we allowed subsidies, e.g., a constant τ_c offset by a labor subsidy $\tau_l = -\tau_c$), that is, government spending is funded by lump sum taxes only:

$$G_t + T_t = 0, \ \forall t$$

- In other words, proportional taxes are distortionary as they affect household's incentives to work and to invest by effectively changing the **relative prices**, distorting the equilibrium outcome

- Question: is $\tau_{lt} > 0$ distortionary when labor supply is inelastic?

Steady state

- To see the effect of distortionary taxes more clearly, consider a time-invariant policy

$$\tau_c, \tau_k, \tau_l, G, T$$

  and focus on the steady state with time invariant c, k, l, r, w

- Household optimization conditions evaluated in eqm become

$$\begin{aligned}\frac{v'(l)}{u'(c)} =& \frac{1-\tau_l}{1+\tau_c} w = \frac{1-\tau_l}{1+\tau_c} F_2\left(\frac{k}{l}, 1\right) \\ \frac{1}{\beta} =& 1 + (1-\tau_k)(r-\delta) = 1 + (1-\tau_k)\left(F_1\left(\frac{k}{l}, 1\right) - \delta\right)\end{aligned}$$

- We can immediately see that k/l depends on τ_k only and strictly decreases in τ_k, and interest rate r strictly increases in τ_k

$$r = F_1\left(\frac{k}{l}, 1\right) = \frac{\beta^{-1} - 1}{1 - \tau_k} + \delta$$

Steady state

- Wage $w = F_2(k/l, 1)$ strictly increases in k/l and so strictly decreases in τ_k if $F_{12} > 0$
- What about labor supply? Whether labor supply increases or decreases in tax rates depend on whether substitution or income effect is stronger

$$\frac{v'(l)}{u'(c)} = \frac{1 - \tau_l}{1 + \tau_c} w$$

- A special case without income effect: quasi-linear utility with $u(c) = c$ and $v(l)$ strictly convex. Then labor supply l strictly decreases in τ_l, τ_c, τ_k, so capital stock k also strictly decrease in these tax rates.
- General equilibrium effect

Government debt

- Now consider government debt in the form of a one-period (discount) bond B_t
- At t, gov issues/sells B_{t+1} at market bond price q_t and repays B_t. Fix $\{\tau_{ct}, \tau_{kt}, \tau_{lt}, G_t, T_t\}_{t=0}^{\infty}$ and let debt B_{t+1} adjust
- Gov budget constraint becomes (without loss, $B_0 = 0$)

$$G_t + T_t + B_t = \tau_{ct} C_t + \tau_{kt}(r_t - \delta) K_t + \tau_{lt} w_t L_t + q_t B_{t+1}$$

- Household problem becomes

$$\max_{\{c_t, l_t, k_{t+1}, b_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t [u(c_t) - v(l_t)]$$

$$\text{s.t.} \quad (1+\tau_{ct}) c_t + k_{t+1} + q_t b_{t+1} \leq (1-\tau_{lt}) w_t l_t + k_t + (1-\tau_{kt})(r_t - \delta) k_t + b_t + T_t, \ \forall t$$

given $b_0 = 0$.

Sequential markets equilibrium

Definition 5

Given a government policy sequence $\{\tau_{ct}, \tau_{kt}, \tau_{lt}, G_t, T_t\}_{t=0}^{\infty}$, a **sequential markets equilibrium** consists of a sequence of prices $\{r_t^*, w_t^*, q_t^*\}_{t=0}^{\infty}$, household allocation $\{c_t^*, l_t^*, k_{t+1}^*, b_{t+1}^*\}_{t=0}^{\infty}$, firm allocation $\{K_t^d, L_t^d\}_{t=0}^{\infty}$, and government debt $\{B_{t+1}^*\}_{t=0}^{\infty}$, such that

- (Household optimization) given $\{r_t^*, w_t^*, q_t^*\}_{t=0}^{\infty}$, household optimizes at $\{c_t^*, l_t^*, k_{t+1}^*, b_{t+1}^*\}_{t=0}^{\infty}$;
- (Firm optimization) given $\{r_t^*, w_t^*\}_{t=0}^{\infty}$, firm optimizes at $\{K_t^d, L_t^d\}_{t=0}^{\infty}$, $\forall t$;
- (Government budget) $\{B_{t+1}^*\}_{t=0}^{\infty}$ balances the gov budget $\forall t$;
- (Market clearing) all markets clear in every period t:

$$c_t^* + k_{t+1}^* + G_t = F(K_t^d, L_t^d) + (1-\delta)k_t^*, \quad k_t^* = K_t^d, \quad l_t^* = L_t^d, \quad b_{t+1}^* = B_{t+1}^*, \forall t.$$

Equilibrium charaterization

- Note the new FOC wrt b_{t+1}:

$$\begin{aligned}\beta^t u'(c_t) =& \lambda_t(1+\tau_{ct})\\ \beta^t v'(l_t) =& \lambda_t(1-\tau_{lt})w_t\\ \lambda_t =& \lambda_{t+1}[1+(1-\tau_{k,t+1})(r_{t+1}-\delta)]\\ q_t\lambda_t =& \lambda_{t+1}\end{aligned}$$

  TVCs: $\lim_{t\to\infty}\lambda_t k_{t+1}=0$, and $\lim_{t\to\infty}\lambda_t b_{t+1}=0$.

- For q_t to be part of an equilibrium, it must satisfy (**no arbitrage**)

$$1 = q_t[1+(1-\tau_{k,t+1})(r_{t+1}-\delta)]$$

  household must be indifferent between investing in physical capital and purchasing gov bond

Consolidating budget constraints

- Define $\widehat{p}_0 := 1$ and $\widehat{p}_t := \prod_{s=0}^{t-1} q_s = \prod_{s=0}^{t-1} \frac{1}{1+(1-\tau_{k,s+1})(r_{s+1}-\delta)}$ for $t \geq 1$. Combining gov balanced budgets up to T by eliminating B_t yields

$$\sum_{t=0}^{T} \widehat{p}_t (G_t + T_t) = \sum_{t=0}^{T} \widehat{p}_t [\tau_{ct} C_t + \tau_{kt}(r_t - \delta) K_t + \tau_{lt} w_t L_t] + \widehat{p}_{T+1} B_{T+1}$$

- Note that household TVC for b_t evaluated in equilibrium ($b_{t+1}^* = B_{t+1}^*$) implies that gov cannot borrow indefinitely, i.e., $\lim_{t\to\infty} \widehat{p}_t B_t = 0$

- So

$$\sum_{t=0}^{\infty} \widehat{p}_t G_t = \sum_{t=0}^{\infty} \widehat{p}_t [\tau_{ct} C_t + \tau_{kt}(r_t - \delta) K_t + \tau_{lt} w_t L_t + (-T_t)]$$

  That is, ultimately, gov spending is funded by tax revenue

Ricardian equivalence

- When TVC for b_t holds, we can consolidate household budget constraints as well

$$\sum_{t=0}^{\infty} \widehat{p}_t[(1+\tau_{ct})c_t + k_{t+1}] = \sum_{t=0}^{\infty} \widehat{p}_t[(1-\tau_{lt})w_t l_t + k_t + (1-\tau_{kt})(r_t - \delta)k_t + T_t]$$

  This is household budget constraint in present value (recall: Arrow-Debreu market structure)

- Household decisions of c_t, l_t, k_{t+1} remain the same with this constraint

- **Ricardian equivalence**: given $\{\tau_{ct}, \tau_{kt}, \tau_{lt}, G_t\}_{t=0}^{\infty}$, **the timing of lump sum taxes $\{T_t\}_{t=0}^{\infty}$ is irrelevant** for equilibrium outcome as long as the PV remains unchanged at $\sum_{t=0}^{\infty} \widehat{p}_t T_t$ and gov can issue bonds: only the debt path $\{B_{t+1}^*\}$ adjusts

Human Capital (optional)

Incorporating human capital into the neoclassical growth model

- A simple (but not great) way to model human capital investment
- Model setup is identical to the simple neoclassical growth model, except for the CRS aggregate production function

$$F(K, HL)$$

and human capital accumulation

$$H_{t+1} = (1 - \delta_H)H_t + I_t^H, \quad \delta_H \in [0, 1].$$

- We make the same **assumptions** on the production function, plus $F_{12} > 0$ (complementary inputs). Normalize $L_t = 1$.
- We begin with a planner's problem. Then we work our way toward a recursive competitive equilibrium.

Social planner's problem in sequential formulation

At $t = 0$, the planner's problem is to find the Pareto efficient allocation:

$$
\begin{aligned}
W(\overline{k}_0, \overline{h}_0) = \max_{\{c_t, i_t^K, i_t^H, k_t, h_t\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t u(c_t) \\
\text{subject to} \quad c_t + i_t^K + i_t^H \leq & F(k_t, h_t), \ \forall t, \\
k_{t+1} = & (1 - \delta_K) k_t + i_t^K, \ \forall t, \\
h_{t+1} = & (1 - \delta_H) h_t + i_t^H, \ \forall t, \\
c_t, k_t, h_t \geq & 0, \ \forall t, \\
k_0 \leq & \overline{k}_0, \text{ and } h_0 \leq \overline{h}_0.
\end{aligned}
$$

Note that both i_t^K and i_t^H, so k_{t+1} and h_{t+1}, are dynamic decisions.

The Euler equations

- It's straightforward to derive the **two** sets of **Euler equations**: $\forall t$,

$$\begin{aligned} u'(c_t) =& \beta u'(c_{t+1})[F_1(k_{t+1}, h_{t+1}) + 1 - \delta_K] \\ u'(c_t) =& \beta u'(c_{t+1})[F_2(k_{t+1}, h_{t+1}) + 1 - \delta_H] \end{aligned}$$

  where $c_t = F(k_t, h_t) + (1 - \delta_K)k_t + (1 - \delta_H)h_t - k_{t+1} - h_{t+1}$

- The two transversality conditions are

$$\begin{aligned} \lim_{t \to \infty} \beta^t u'(c_t) k_{t+1} =& 0 \\ \lim_{t \to \infty} \beta^t u'(c_t) h_{t+1} =& 0 \end{aligned}$$

Physical and human capital

- Observe that the Euler equations imply that

$$F_1(k_{t+1}, h_{t+1}) - \delta_K = F_2(k_{t+1}, h_{t+1}) - \delta_H, \quad \forall t$$

  interpretation: the marginal returns to both investments must be the same

- CRS production function implies that F_1 and F_2 are homogenous of degree 0. It follows that optimal physical capital and optimal human capital are proportional:

$$h_{t+1} = \xi k_{t+1}, \quad \text{for some } \xi > 0$$

  so $F(k_{t+1}, h_{t+1}) = F(1, \xi)k_{t+1}$, $F_1(k_{t+1}, h_{t+1}) = F_1(1, \xi)$, $F_2(k_{t+1}, h_{t+1}) = F_2(1, \xi)$

- Whatever $(\overline{k}_0, \overline{h}_0)$, the planner jumps to the ratio ξ at $t = 1$, which may require $i_0^K < 0$ or $i_0^H < 0$: fine here, but not with irreversible investment!

Consolidating the Euler equations and TVCs

- We obtain a system of Euler equations very similar to the basic NGM

$$u'(c_t) = \beta u'(c_{t+1})[F_1(1,\xi) + 1 - \delta_K]$$

where $c_t = [F(1,\xi) + 1 - \delta_K + (1-\delta_H)\xi]k_t - (1+\xi)k_{t+1}$, $\forall t > 0$, and $c_0 = F(\overline{k}_0, \overline{h}_0) + (1-\delta_K)\overline{k}_0 + (1-\delta_H)\overline{h}_0 - (1+\xi)k_1$

- And one TVC is sufficient:

$$\lim_{t\to\infty} \beta^t u'(c_t)k_{t+1} = 0$$

Steady state and BGP?

- Let $R := F_1(1, \xi) + 1 - \delta_K = F_2(1, \xi) + 1 - \delta_H$ be the (constant!) gross return on either capital: this is an AK model
- By Euler's theorem, $F(1, \xi) = F_1(1, \xi) + F_2(1, \xi)\xi$, so net output per unit of k is

$$F(1, \xi) - \delta_K - \delta_H \xi = (R - 1)(1 + \xi)$$

- Does a steady state exist? The Euler equation $u'(c_t) = \beta R\, u'(c_{t+1})$ says it depends on βR vs 1
  - if $\beta R = 1$: $c' = c$, and any k with $h = \xi k$ is a steady state ($c > 0$ since $R = 1/\beta > 1$)
  - if $\beta R < 1$: $c' < c$, no long-run growth
  - if $\beta R > 1$: $c' > c$, long-run growth

Steady state and BGP?

- Suppose $\beta R > 1$ and assume CRRA utility $u(c) = \dfrac{c^{1-\sigma} - 1}{1-\sigma}$, $\sigma > 0$
- BGP requires that c, k, h grow at the same constant **endogenous** rate γ s.t.

$$c_t^{-\sigma} = \beta((1+\gamma)c_t)^{-\sigma} R \quad \Longrightarrow \quad 1 + \gamma = (\beta R)^{\frac{1}{\sigma}}$$

- To ensure $c_t > 0$, we need $F(1, \xi) - \delta_K - \delta_H \xi > \gamma(1 + \xi)$, i.e., $1 + \gamma < R$
- To ensure TVC holds, we need $\beta(1+\gamma)^{1-\sigma} < 1$. Since $\beta(1+\gamma)^{-\sigma} = 1/R$, this is again $1 + \gamma < R$
- One condition, $1 + \gamma < R \iff \beta^{1/\sigma} R^{1/\sigma - 1} < 1$: the same as the finite-PV condition in the consumption-saving example, with $1 + r = R$

Social planner's problem in recursive formulation

- State variables: k, h
- Bellman equation:

$$V(k,h) = \max_{k',h' \geq 0} \{u(c) + \beta V(k',h')\}$$
$$\text{where } c = F(k,h) + (1-\delta_K)k + (1-\delta_H)h - k' - h' \geq 0$$

- Our assumptions ensure that V is strictly concave and policies are single-valued function. Let $g_K(k,h)$ and $g_H(k,h)$ be the associated policy functions.

The functional Euler equations

- FOCs at $k' = g_K(k,h)$ and $h' = g_H(k,h)$

$$\begin{aligned} u'(c) =& \beta V_1(k',h') \\ u'(c) =& \beta V_2(k',h') \end{aligned}$$

- The envelope conditions

$$\begin{aligned} V_1(k,h) =& u'(c)[F_1(k,h) + 1 - \delta_K] \\ V_2(k,h) =& u'(c)[F_2(k,h) + 1 - \delta_H] \end{aligned}$$

- The Euler equations at $k' = g_K(k,h)$ and $h' = g_H(k,h)$

$$\begin{aligned} u'(c) =& \beta u'(c')[F_1(k',h') + 1 - \delta_K] \\ u'(c) =& \beta u'(c')[F_2(k',h') + 1 - \delta_H] \end{aligned}$$

Consolidating the Euler equations

- It follows that

$$h' = \xi k'$$

  or

$$g_H(k,h) = \xi g_K(k,h)$$

- So the Euler equations become

$$\begin{aligned} &u'(F(k,h) + (1-\delta_K)k + (1-\delta_H)h - (1+\xi)k') \\ =&\beta u'\big([F(1,\xi) + (1-\delta_K) + (1-\delta_H)\xi]k' - (1+\xi)k''\big)[F_1(1,\xi) + 1 - \delta_K] \end{aligned}$$

  where $k' = g_K(k,h)$ and $k'' = g_K(k',h') = g_K(g_K(k,h), \xi g_K(k,h))$

- Note that we cannot replace h with ξk

Human Capital: Competitive Equilibrium (optional)

Sequential markets

- Households own both kinds of capital, labor, and all firms
- Competitive markets for time-t commodities open at t: consumption good, capital service, human capital augmented labor service with corresponding prices

$$1, r_t, w_t$$

- Firm problem remains static: given r_t, w_t at t,

$$\max_{K,N>0} F(K,N) - r_t K - w_t N$$

where N is the effective labor input

Sequential markets

- Household problem at $t = 0$: given $\{r_t, w_t\}_{t=0}^{\infty}$ (for simplicity, we drop π_t as we know $\pi_t = 0$)

$$\begin{aligned}
&\max_{\{c_t, i_t^K, i_t^H, k_t, h_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
\text{subject to} \quad & c_t + i_t^K + i_t^H \leq r_t k_t + w_t h_t, \ \forall t \\
& k_{t+1} = (1 - \delta_K) k_t + i_t^K, \ \forall t, \\
& h_{t+1} = (1 - \delta_H) h_t + i_t^H, \ \forall t, \\
& c_t, k_t, h_t \geq 0, \ \forall t, \\
& k_0 \leq \overline{k}_0, \text{ and } h_0 \leq \overline{h}_0.
\end{aligned}$$

Sequential markets equilibrium

Definition 6
A **sequential markets equilibrium** consists of a sequence of prices $\{r_t^*, w_t^*\}_{t=0}^{\infty}$, household allocation $\{c_t^*, i_t^{K*}, i_t^{H*}, k_t^*, h_t^*\}_{t=0}^{\infty}$ given $\overline{k}_0$ and $\overline{h}_0$, and firm allocation $\{K_t^d, N_t^d\}_{t=0}^{\infty}$, such that

- (Household optimization) given $\{r_t^*, w_t^*\}_{t=0}^{\infty}$, $\{c_t^*, i_t^{K*}, i_t^{H*}, k_t^*, h_t^*\}_{t=0}^{\infty}$ solves the household's problem;
- (Firm optimization) given $\{r_t^*, w_t^*\}_{t=0}^{\infty}$, $\{K_t^d, N_t^d\}_{t=0}^{\infty}$ solves the firm's per-period problem, $\forall t$;
- (Market clearing) all markets clear in every period t:

$$c_t^* + i_t^{K*} + i_t^{H*} = F(K_t^d, N_t^d), \qquad k_t^* = K_t^d, \qquad h_t^* = N_t^d.$$

Characterization of SME

- FOCs of firm optimization at t

$$r_t = F_1(K_t, N_t), \quad w_t = F_2(K_t, N_t)$$

evaluated at the SME

$$r_t^* = F_1(k_t^*, h_t^*), \quad w_t^* = F_2(k_t^*, h_t^*)$$

Characterization of SME

- Household optimization implies the same two sets of Euler equations

$$\begin{aligned} u'(c_t) =& \beta u'(c_{t+1})[r_{t+1} + 1 - \delta_K] \\ u'(c_t) =& \beta u'(c_{t+1})[w_{t+1} + 1 - \delta_H] \end{aligned}$$

  where $c_t = r_t k_t + w_t h_t + (1 - \delta_K)k_t + (1 - \delta_H)h_t - k_{t+1} - h_{t+1}$

- The two transversality conditions are

$$\lim_{t\to\infty} \left(\prod_{s=0}^{t-1} \frac{1}{r_{s+1} + 1 - \delta_K} \right) k_{t+1} = 0$$

$$\lim_{t\to\infty} \left(\prod_{s=0}^{t-1} \frac{1}{w_{s+1} + 1 - \delta_H} \right) h_{t+1} = 0$$

- We can verify that SME allocation coincides with the planner's choice

Recursive formulation of household problem

- State variables

$$\underbrace{k, h}_{\text{individual state}} \quad , \quad \underbrace{K, H}_{\text{aggregate state}}$$

- Pricing equations

$$r(K, H), \quad w(K, H)$$

- Aggregate laws of motion

$$K' = G^K(K, H), \quad H' = G^H(K, H)$$

Recursive formulation of household problem

The Bellman equation for the household problem can be written as

$$V(k,h,K,H) = \max_{k',h' \geq 0} \left\{ u(c) + \beta V(k',h',G^K(K,H),G^H(K,H)) \right\}$$
$$\text{where } c = r(K,H)k + w(K,H)h + (1-\delta_K)k + (1-\delta_H)h - k' - h' \geq 0$$

The associated policy functions are

$$k' = g^K(k,h,K,H), \quad h' = g^H(k,h,K,H)$$

Recursive competitive equilibrium

Definition 7

A **recursive competitive equilibrium** consists of household value function $V : \mathbb{R}_+^4 \to \mathbb{R}$ and policy functions $g^K, g^H : \mathbb{R}_+^4 \to \mathbb{R}_+$, **pricing functions** $r, w : \mathbb{R}_+^2 \to \mathbb{R}_+$, and the aggregate laws of motion $G^K, G^H : \mathbb{R}_+^2 \to \mathbb{R}_+$ such that

- (household optimization) given r, w and G^K, G^H, the solution to the household's Bellman equation is V with associated policies g^K, g^H;
- (firm optimization) r, w satisfy $r(K, H) = F_1(K, H)$ and $w(K, H) = F_2(K, H)$, $\forall (K, H)$;
- (**consistency**) individual decision is consistent with aggregate law of motion

$$g^K(K, H, K, H) = G^K(K, H), \quad g^H(K, H, K, H) = G^H(K, H), \ \forall K, H.$$

Straightforward to verify that market clearing conditions hold.

Appendix: CRRA utility and the EIS

back

- **Elasticity of intertemporal substitution**: how much c_{t+1}/c_t responds to its relative price $1+r$,

$$\text{EIS} := \frac{d\ln(c_{t+1}/c_t)}{d\ln(1+r)} \quad \text{along the Euler equation } u'(c_t) = \beta(1+r)u'(c_{t+1})$$

- CRRA, $u'(c) = c^{-\sigma}$: the Euler equation is $(c_{t+1}/c_t)^\sigma = \beta(1+r)$, so

$$\ln\frac{c_{t+1}}{c_t} = \frac{1}{\sigma}\big[\ln\beta + \ln(1+r)\big] \quad \Longrightarrow \quad \text{EIS} = \frac{1}{\sigma}$$

- General u: take logs of the Euler equation and differentiate at $c_t = c_{t+1} = c$,

$$\frac{cu''(c)}{u'(c)}\left(d\ln c_t - d\ln c_{t+1}\right) = d\ln(1+r) \quad \Longrightarrow \quad \text{EIS} = -\frac{u'(c)}{c\,u''(c)}$$

Appendix: CRRA utility and relative risk aversion

back

- **Relative risk aversion** (Arrow–Pratt): $\mathrm{RRA}(c) := -\dfrac{c\,u''(c)}{u'(c)}$
- Why: a small gamble $c(1+\varepsilon)$, $\mathbb{E}\varepsilon = 0$, $\mathrm{Var}(\varepsilon) = s^2$. The share π of c the household would pay to avoid it solves

$$u\big(c(1-\pi)\big) = \mathbb{E}\big[u\big(c(1+\varepsilon)\big)\big] \overset{\text{Taylor}}{\Longrightarrow} -u'(c)\,c\,\pi \approx \tfrac{1}{2}u''(c)\,c^2 s^2 \quad \Longrightarrow \quad \pi \approx \tfrac{1}{2}\,\mathrm{RRA}(c)\,s^2$$

- CRRA: $u''(c) = -\sigma c^{-\sigma-1}$, so $\mathrm{RRA} = \sigma$ for all c: *constant* relative risk aversion
- Comparing with the previous slide: with time-separable expected utility, $\mathrm{EIS} = 1/\mathrm{RRA}$ locally, for any u. Epstein–Zin preferences separate the two