ECON 720: Advanced Macroeconomic Theory I

Competitive Equilibrium Growth

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Competitive equilibrium growth

- Recall the model setup
- **Social planner's problem** yields the efficient allocation $\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$
- Now: What **prices** and **allocation** emerge when households and firms make their own decisions and interact in **(competitive) markets**?
- Need to specify: firm ownership, market structures and equilibrium concepts
- Easiest to think of a representative agent as one out of a unit-measure continuum of identical agents making identical decisions.

Competitive equilibrium growth

We assume **households own firms** and consider **perfect competition** in all markets, but with different timing assumptions:

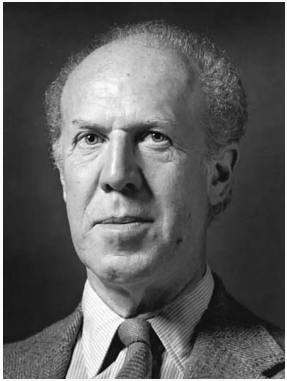
- Time-0 trading and an **Arrow-Debreu equilibrium (ADE)**
  - closest to micro theory and the planner's (SP): all markets are open at $t = 0$ and all decisions are made at $t = 0$
- Sequential trading and a **sequential markets equilibrium (SME)**
  - most natural and intuitive: markets for time-t commodities open at time t
- Sequential trading and a **recursive competitive equilibrium (RCE)**
  - arguably most natural and practical: recursive formulation of household problem

All are competitive equilibria. Same equilibrium allocations in "standard" cases with complete markets.

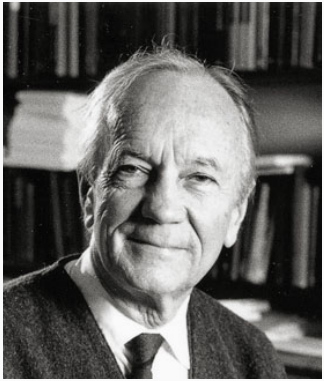
General Equilibrium



Kenneth Arrow



Gérard Debreu



Lionel McKenzie

Arrow-Debreu Equilibrium

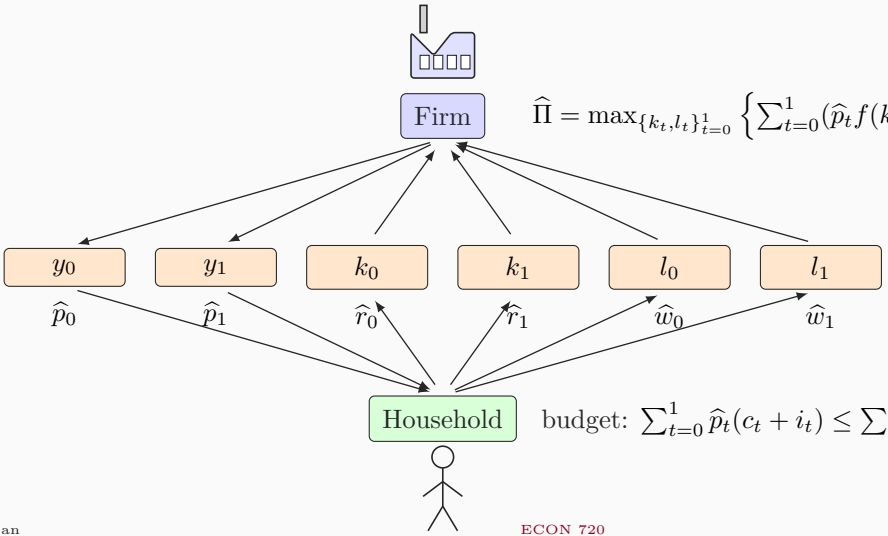
Time-0 trading

- All markets are open at $t = 0$ only, including
  - a market for period-t consumption good, with price $\widehat{p}_t$, $\forall t$
  - a market for period-t capital service, with price/rental rate $\widehat{r}_t$, $\forall t$
  - a market for period-t labor service, with wage rate $\widehat{w}_t$, $\forall t$

  e.g., with a finite horizon $t = 0, 1, \ldots, T$, all $3(T + 1)$ markets are open at $t = 0$
- Imagine all agents trade "claims" of each commodity in each period t
- The same good in different periods are considered different commodities, traded in different markets, and can have different prices

Time-0 trading

A 2-period example of Arrow-Debreu market structure. At $t = 0$:



A representative firm's problem

It is easier to start with a representative firm's problem.

- At $t = 0$, when all markets are open, the firm sees and **takes as given all prices** $\{\widehat{p}_t, \widehat{r}_t, \widehat{w}_t\}_{t=0}^{\infty}$ to maximize profit

$$\widehat{\Pi} = \max_{\{K_t, L_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} (\widehat{p}_t F(K_t, L_t) - \widehat{r}_t K_t - \widehat{w}_t L_t)$$

  which is equivalent to solving a sequence of static problems, such that $\forall t$,

$$\widehat{\pi}_t = \max_{K_t, L_t} \widehat{p}_t F(K_t, L_t) - \widehat{r}_t K_t - \widehat{w}_t L_t$$

  and $\widehat{\Pi} = \sum_{t=0}^{\infty} \widehat{\pi}_t$.

- No decisions are made from $t = 1$ onward

A representative household's problem

At $t = 0$, when all markets are open, the household with $\overline{k}_0$ sees and **takes as given all prices** $\{\widehat{p}_t, \widehat{r}_t, \widehat{w}_t\}_{t=0}^{\infty}$ and profit $\widehat{\Pi}$ to maximize lifetime utility

$$\max_{\{c_t, i_t, l_t, k_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{subject to} \quad \sum_{t=0}^{\infty} \widehat{p}_t(c_t + i_t) \leq \sum_{t=0}^{\infty} (\widehat{r}_t k_t + \widehat{w}_t l_t) + \widehat{\Pi}, \qquad \text{(lifetime budget constraint)}$$

$$k_{t+1} = (1-\delta)k_t + i_t, \ \forall t, \qquad \text{(capital accumulation)}$$

$$c_t \geq 0, l_t \in [0,1], k_t \geq 0, \ \forall t, \text{ and } k_0 \leq \overline{k}_0. \qquad \text{(feasibility)}$$

Notes: **Only 1 budget constraint (all trades at $t = 0$)**. No decisions after $t = 0$.

Equilibrium definition

Definition 1
An **Arrow-Debreu equilibrium** consists of a sequence of prices $\{\widehat{p}_t^*, \widehat{r}_t^*, \widehat{w}_t^*\}_{t=0}^{\infty}$, time-0 profit $\widehat{\Pi}$, household allocation $\{c_t^*, i_t^*, l_t^*, k_t^*\}_{t=0}^{\infty}$ given $\overline{k}_0$, and firm allocation $\{K_t^d, L_t^d\}_{t=0}^{\infty}$, such that

- (Household optimization) given $\{\widehat{p}_t^*, \widehat{r}_t^*, \widehat{w}_t^*\}_{t=0}^{\infty}$ and $\widehat{\Pi}$, $\{c_t^*, i_t^*, l_t^*, k_t^*\}_{t=0}^{\infty}$ solves the household's problem;
- (Firm optimization) given $\{\widehat{p}_t^*, \widehat{r}_t^*, \widehat{w}_t^*\}_{t=0}^{\infty}$, $\{K_t^d, L_t^d\}_{t=0}^{\infty}$ solves the firm's problem such that the optimized total profit evaluated at $t = 0$ is $\widehat{\Pi}$;
- (Market clearing) all markets opening at $t = 0$ clear:

$$c_t^* + i_t^* = F(K_t^d, L_t^d), \qquad k_t^* = K_t^d, \qquad l_t^* = L_t^d, \qquad \forall t.$$

Some observations

- Indeterminant absolute price levels: $\{\widehat{p}_t, \widehat{r}_t, \widehat{w}_t\}_{t=0}^{\infty}$ and $\{a\widehat{p}_t, a\widehat{r}_t, a\widehat{w}_t\}_{t=0}^{\infty}$ lead to the same firm problem or household problem, $\forall a > 0$. **Only relative prices matter.**

- Let time-0 consumption good be the numeraire. Write the sequence of relative good prices as $\{p_t\}_{t=0}^{\infty}$, where $p_t := \widehat{p}_t/\widehat{p}_0$: price of time-$t$ good in the unit of time-0 good.

- Rewrite time-t interest rate, wage, and profit in the unit of time-t good: $r_t := \widehat{r}_t/\widehat{p}_t$ and $w_t := \widehat{w}_t/\widehat{p}_t$, $\pi_t := \widehat{\pi}_t/\widehat{p}_t$

- Then firm problem becomes the following sequence of static decisions: $\forall t$,

$$\max_{K_t, L_t \geq 0} F(K_t, L_t) - r_t K_t - w_t L_t$$

  and household budget constraint becomes ($\Pi := \widehat{\Pi}/\widehat{p}_0 = \sum_{t=0}^{\infty} p_t \pi_t$)

$$\sum_{t=0}^{\infty} p_t(c_t + i_t) \leq \sum_{t=0}^{\infty} p_t(r_t k_t + w_t l_t) + \Pi$$

Redefine equilibrium

Definition 2
An **Arrow-Debreu equilibrium** consists of a sequence of prices $\{p_t^*, p_t^* r_t^*, p_t^* w_t^*\}_{t=0}^{\infty}$ with $p_0^* = 1$, time-0 profit Π, household allocation $\{c_t^*, i_t^*, l_t^*, k_t^*\}_{t=0}^{\infty}$ given k_0, and firm allocation $\{K_t^d, L_t^d\}_{t=0}^{\infty}$, such that

- (Household optimization) given $\{p_t^*, p_t^* r_t^*, p_t^* w_t^*\}_{t=0}^{\infty}$ and Π, $\{c_t^*, i_t^*, l_t^*, k_t^*\}_{t=0}^{\infty}$ solves the household's problem;
- (Firm optimization) given $\{p_t^*, p_t^* r_t^*, p_t^* w_t^*\}_{t=0}^{\infty}$, $\{K_t^d, L_t^d\}_{t=0}^{\infty}$ solves the firm's problem such that the optimized total profit evaluated at $t = 0$ is Π;
- (Market clearing) all markets opening at $t = 0$ clear:

$$c_t^* + i_t^* = F(K_t^d, L_t^d), \qquad k_t^* = K_t^d, \qquad l_t^* = L_t^d, \qquad \forall t.$$

Analysis of firm problem

- Given t-prices denominated in time-0 consumption good $p_t, p_t w_t, p_t r_t$, $\forall t$

$$\pi_t = \max_{K_t, L_t \geq 0} F(K_t, L_t) - r_t K_t - w_t L_t$$

  with FOCs

$$F_1(K_t, L_t) = r_t, \qquad F_2(K_t, L_t) = w_t.$$

- CRS implies $\pi_t = 0$, $\forall t$, so total profit at $t = 0$ is 0:

$$\Pi = \sum_{t=0}^{\infty} p_t \pi_t = 0$$

- Only K/L ratio can be pinned down b/c F_1 and F_2 are homogenous of degree 0:

$$F_1(K_t, L_t) = F_1(K_t/L_t, 1) = r_t, \qquad F_2(K_t, L_t) = F_2(K_t/L_t, 1) = w_t.$$

Analysis of household problem

- Household sets $k_0^* = \overline{k}_0$ and $l_t^* = 1$, $\forall t$, as u is increasing in c_t and independent of l_t
- Replace $i_t = k_{t+1} - (1-\delta)k_t$
- Given price sequence $\{p_t, p_t r_t, p_t w_t\}_{t=0}^{\infty}$ (with $p_0 = 1$) and $\Pi = 0$, rewrite and simplify household problem as

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{subject to} \quad \sum_{t=0}^{\infty} p_t(c_t + k_{t+1} - (1-\delta)k_t) \leq \sum_{t=0}^{\infty} p_t(r_t k_t + w_t)$$

$$k_{t+1} \geq 0, \ \forall t, \quad \text{given } \overline{k}_0.$$

Analysis of household problem

- Let λ be the Lagrangian multiplier for the budget constraint and μ_t for $k_{t+1} \geq 0$

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda \left[\sum_{t=0}^{\infty} p_t(r_t k_t + w_t) - \sum_{t=0}^{\infty} p_t(c_t + k_{t+1} - (1-\delta)k_t) \right] + \sum_{t=0}^{\infty} \mu_t k_{t+1}$$

- FOCs wrt c_t and k_{t+1} imply

$$\begin{aligned} \beta^t u'(c_t) &= \lambda p_t \\ \lambda \Big(p_t - p_{t+1}(r_{t+1} + 1 - \delta) \Big) &= \mu_t \end{aligned}$$

- Observe: FOC wrt k_{t+1} is essentially a restriction on price sequence:

$$p_t \geq p_{t+1}(r_{t+1} + 1 - \delta)$$

Analysis of household problem

- We can't claim $\mu_t = 0$, i.e., $k_{t+1} > 0$, based on household problem alone
- However, we argue the following: $k_{t+1} = 0$ will never be part of a well-defined equilibrium because that would mean $r_{t+1} = F_1(0, 1) = \infty$ (Inada conditions), not a well-defined price.
- So we proceed by setting $\mu_t = 0$ as we aim to characterize the equilibrium, so an equilibrium price sequence must satisfy

$$p_t = p_{t+1}(r_{t+1} + 1 - \delta) \quad \iff \quad \frac{p_{t+1}}{p_t} = \frac{1}{r_{t+1} + 1 - \delta}$$

Analysis of household problem

- From the FOCs, we have the marginal rate of substitution

$$\frac{\beta u'(c_{t+1})}{u'(c_t)} = \frac{p_{t+1}}{p_t}, \quad \text{in particular,} \quad \frac{\beta^t u'(c_t)}{u'(c_0)} = \frac{p_t}{p_0} = p_t$$

  and the **Euler equations** for the household's decision

$$u'(c_t) = \beta u'(c_{t+1})(r_{t+1} + 1 - \delta), \quad \forall t$$

- The transversality condition (TVC) is not a requirement of optimality but a consequence. It can be written as

$$\lim_{t \to \infty} p_t k_{t+1} = 0$$

Equilibrium characterization: allocation

- When all markets clear, we have $K_t^d = k_t^*$ and $L_t^d = 1$, so $F(k_t^*, 1) = f(k_t^*)$ and

$$c_t^* = f(k_t^*) + (1-\delta)k_t^* - k_{t+1}^*, \ \forall t$$

and

$$r_t^* = F_1(k_t^*, 1) = f'(k_t^*), \ \forall t$$

- In equilibrium, the Euler equation must satisfy

$$u'(f(k_t^*) + (1-\delta)k_t^* - k_{t+1}^*) = \beta u'(f(k_{t+1}^*) + (1-\delta)k_{t+1}^* - k_{t+2}^*)(f'(k_{t+1}^*) + 1 - \delta), \quad \forall t$$

which coincides with the social planner's.

Equilibrium characterization: allocation

- What about the TVC? Evaluated at the equilibrium

$$\lim_{t\to\infty} p_t^* k_{t+1}^* = \frac{1}{\lambda} \lim_{t\to\infty} \beta^t u'(c_t^*) k_{t+1}^* = \frac{1}{\lambda} \lim_{t\to\infty} \beta^t u'(f(k_t^*) + (1-\delta)k_t^* - k_{t+1}^*) k_{t+1}^* = 0$$

  which is equivalent to the social planner's TVC.

- Therefore, the allocation at the **Arrow-Debreu equilibrium** coincides with what is **Pareto optimal**.

- This is not a coincidence!

Equilibrium characterization: prices

- ADE prices must satisfy

$$\begin{aligned} r_t^* =& F_1(k_t^*, 1) = f'(k_t^*), \ \forall t \\ w_t^* =& F_2(k_t^*, 1) = f(k_t^*) - k_t^* f'(k_t^*), \ \forall t \\ p_t^* =& \frac{\beta^t u'(c_t^*)}{u'(c_0^*)} = \prod_{s=0}^{t-1} \frac{1}{r_{s+1}^* + 1 - \delta}, \ \forall t \geq 1 \end{aligned}$$

- All markets must clear simultaneously: note how "everything affects everything"
- How do we think about p_t^*? **An intertemporal exchange rate or the price of time**: we need to give up p_t^* units of consumption good in 0 in exchange for 1 unit of consumption good in t

Taking stock

What can we say about the Arrow-Debreu equilibrium?

- We can characterize it, i.e., find the set of equations (EEs and TVC) that the equilibrium prices and allocation must satisfy, but we generally can't solve it directly
- The ADE **allocation** coincides with what is Pareto **efficient**
  - So we can "solve" for an ADE using Negishi approach: solve the planner's problem to get the allocation, then compute the prices.

Is there a corresponding recursive formulation?

- NO
- Non-stationary: no markets from $t = 1$ onward, so no recursive structure

Sequential Markets Equilibrium

Sequential trading

- Markets for period-t commodities open in period t, $\forall t$
  - period-t consumption good, with normalized price 1
  - period-t capital service, with price r_t in the unit of period-t good
  - period-t labor service, with price w_t in the unit of period-t good
- Why can we normalize the good price in every period to be 1, i.e., why do we have a numeraire every period?

Firm and household problems

- In every period t, given prices in this period r_t, w_t, firm maximizes per-period profit

$$\pi_t = \max_{K_t, L_t \geq 0} F(K_t, L_t) - r_t K_t - w_t L_t$$

- At $t = 0$, the household still maximizes the lifetime utility, knowing markets will open period by period, and taking as given all prices and profits $\{r_t, w_t, \pi_t\}_{t=0}^{\infty}$

$$\begin{aligned} \max_{\{c_t, i_t, l_t, k_t\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{subject to} \quad c_t + i_t \leq & r_t k_t + w_t l_t + \pi_t, \ \forall t \\ k_{t+1} = & (1-\delta) k_t + i_t, \ \forall t \\ c_t \geq & 0, \ \forall t \\ l_t \in & [0, 1], \ \forall t \\ k_t \geq & 0, \ \forall t, \ \text{with } k_0 \leq \overline{k}_0. \end{aligned}$$

A budget constraint in each period t: expenditure cannot exceed income in any period.

Equilibrium definition

Definition 3
A **sequential markets equilibrium** consists of a sequence of prices $\{r_t^*, w_t^*\}_{t=0}^{\infty}$, per-period profit $\{\pi_t^*\}_{t=0}^{\infty}$, household allocation $\{c_t^*, i_t^*, l_t^*, k_t^*\}_{t=0}^{\infty}$ given k_0, and firm allocation $\{K_t^d, L_t^d\}_{t=0}^{\infty}$, such that

- (Household optimization) given $\{r_t^*, w_t^*\}_{t=0}^{\infty}$ and $\{\pi_t^*\}_{t=0}^{\infty}$, $\{c_t^*, i_t^*, l_t^*, k_t^*\}_{t=0}^{\infty}$ solves the household's problem;
- (Firm optimization) given $\{r_t^*, w_t^*\}_{t=0}^{\infty}$, $\{K_t^d, L_t^d\}_{t=0}^{\infty}$ solves the firm's per-period problem such that period-t profit is π_t^*, $\forall t$;
- (Market clearing) all markets clear in every period t:

$$c_t^* + i_t^* = F(K_t^d, L_t^d), \qquad k_t^* = K_t^d, \qquad l_t^* = L_t^d, \qquad \forall t.$$

Analysis of household problem

- Firm problem is identical. Omitted.
- Given $\{r_t, w_t\}_{t=0}^{\infty}$ and $\pi_t = 0$, $\forall t$, household problem can be written as

$$\begin{aligned} &\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{subject to} \quad & c_t + k_{t+1} \leq (r_t + 1 - \delta) k_t + w_t, \ \forall t \\ & k_{t+1} \geq 0, \ \forall t, \quad \text{given } \overline{k}_0. \end{aligned}$$

- Let λ_t be the Lagrangian multiplier for the budget constraint in period t (and ignore the constraint $k_{t+1} \geq 0$ as we know it won't be binding in equilibrium)

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t \Big[(r_t + 1 - \delta) k_t + w_t - c_t - k_{t+1} \Big]$$

Analysis of household problem

- FOCs wrt c_t and k_{t+1} imply

$$\begin{aligned}\beta^t u'(c_t) &= \lambda_t \\ \lambda_t &= \lambda_{t+1}(r_{t+1} + 1 - \delta)\end{aligned}$$

- So we have the Euler equations

$$u'(c_t) = \beta u'(c_{t+1})(r_{t+1} + 1 - \delta), \quad \forall t$$

- The transversality condition (TVC) now can be written as

$$\lim_{t\to\infty} \lambda_t k_{t+1} = \lambda_0 \lim_{t\to\infty} \left(\prod_{s=0}^{t-1} \frac{1}{r_{s+1} + 1 - \delta} \right) k_{t+1} = 0$$

Equilibrium characterization

- It is straightforward to verify that SME outcome is once again efficient
  - Household's Euler equations and TVC evaluated at SME coincide with the planner's
- SME prices are equivalent to the ADE prices

$$\begin{aligned} r_t^* &= F_1(k_t^*, 1) = f'(k_t^*) \\ w_t^* &= F_2(k_t^*, 1) = f(k_t^*) - k_t^* f'(k_t^*) \end{aligned}$$

- This equivalence is **NOT** a general result!

ADE vs. SME

Equivalent allocation with **representative/identical** households. In general:

- Arrow-Debreu market structure: a sense of "market completeness"
  - all goods at all times are priced and traded simultaneously
  - one lifetime budget constraint: allocate purchasing power freely **across time**
- Sequential markets: not as much freedom, not always complete
  - only a subset of goods are priced and traded each time
  - period-by-period constraint: how to transfer purchasing power across time?
- Equivalence? If enough (financial) assets available for sequential markets
  - financial assets: means to transfer purchasing power (NOT available resource) across time

Taking stock

What can we say about the sequential markets equilibrium?

- Same things as we can say about ADE

Is there a corresponding recursive formulation?

- YES
- Stationary: same set of markets open every period, suggesting a possible recursive structure

Recursive Competitive Equilibrium

Recursive competitive equilibrium

We want to find a recursive counterpart to the SME

- Dynamic household optimization problem presented as a **Bellman equation**
- Equilibrium is defined on value and policy and other **functions**

Why?

- This is often the only way to "solve" for an infinite-horizon equilibrium
- Especially when the welfare theorems do not hold (so no Negishi shortcut)

However

- Less straightforward than the planner's case

Two things to deal with

1. To write all dynamic optimization problems recursively as **Bellman equations**: only household problem is dynamic in this model

2. To define and solve for the equilibrium accordingly

We start from a fully stationary environment: a steady state where prices and allocation are constant over time, $r_t = r$ and $w_t = w$, $\forall t$

Recursive formulation of household problem with constant prices

Given constant r, w:

- Firm problem remains static: $\pi = 0 = \max_{K,L} F(K, L) - rK - wL$
- Household problem (after simplification) in sequential formuation

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u((r + 1 - \delta)k_t + w - k_{t+1}) \quad \text{subject to } k_{t+1} \in [0, (r + 1 - \delta)k_t + w]$$

- In recursive formulation: Bellman equation

$$V(k) = \max_{k' \in [0,(r+1-\delta)k+w]} \{u((r + 1 - \delta)k + w - k') + \beta V(k')\}$$

Recursive formulation of household problem with constant prices

Bellman equation:

$$V(k) = \max_{k' \in [0, (r+1-\delta)k+w]} \{u((r+1-\delta)k + w - k') + \beta V(k')\}$$

- State variable (whatever the decision-maker needs to know to make the decision)

  own capital stock this period: k

  we don't need to include r, w when they are time-invariant

- Control variable (the choice)

  own capital stock next period: k'

Recursive formulation of household problem with constant prices

- It's not hard to verify that the Bellman operator is a contraction mapping and preserves the properties of u
- Strict concavity of u ensures that a single-valued policy function exists, denoted as g, such that

$$V(k) = u((r + 1 - \delta)k + w - g(k)) + \beta V(g(k)).$$

Equilibrium definition

Definition 4
A **recursive steady-state competitive equilibrium** consists of scalar prices $r, w > 0$, household value function $V : \mathbb{R}_+ \to \mathbb{R}$ and policy functions $g : \mathbb{R}_+ \to \mathbb{R}_+$, and aggregate capital stock $K > 0$ such that

- (household optimization) given r, w, the solution to the household's Bellman equation is V with an associated policy g;
- (firm optimization) given r, w, firm optimizes at K and $L = 1$, i.e., $r = F_1(K, 1)$ and $w = F_2(K, 1)$;
- (**consistency**) individual decision is consistent with aggregate outcome

$$g(K) = K.$$

Consistency and market clearing conditions

- The new **consistency** condition requires that the aggregate outcome is induced by and thus compatible with individual decisions
- That is, if every household owns $k = K$ capital stock at t, then its optimal capital stock at $t+1$ is $k' = g(K) = K' = K$ at a steady state
- Where are the market clearing conditions?
  - We already impose capital and labor market clearing conditions
  - That for consumption good market is redundant when households own firms and when factor markets clear:

$$c + i = \underbrace{(r + 1 - \delta)K + w + \pi - g(K)}_{c} + \underbrace{g(K) - (1 - \delta)K}_{i} = rK + w + \pi = F(K, 1)$$

Steady-state equilibrium characterization

- FOC of household problem (ignoring constraints) for $k' = g(k)$

$$u'((r+1-\delta)k + w - k') = \beta V'(k')$$

- Envelope condition at k with $k' = g(k)$

$$V'(k) = u'((r+1-\delta)k + w - k')(r+1-\delta)$$

- The Euler equation when $k' = g(k)$ and $k'' = g(g(k))$

$$u'((r+1-\delta)k + w - k') = \beta u'((r+1-\delta)k' + w - k'')(r+1-\delta)$$

Steady-state equilibrium characterization

- Evaluate the Euler equation at the steady-state equilibrium with

$$\begin{aligned} r =& F_1(K, 1) = f'(K) \\ w =& F_2(K, 1) = f(K) - Kf'(K) \\ k =& K = g(K) = g(g(K)) \end{aligned}$$

  observe how prices r and w can be viewed as functions of the aggregate K

- We obtain the same "modified golden rule" as in the planner's steady state:

$$\begin{aligned} 1 =& \beta(r + 1 - \delta) \\ f'(K) =& \frac{1}{\beta} - 1 + \delta \end{aligned}$$

  we can solve for the equilibrium K and r, and the rest of the equilibrium follows

- We know that the economy is at its exact steady state only when $\overline{k}_0 = K$

More general case

- Outside of a steady state, equilibrium prices and allocation change over time
- In the sequential formulation, household knows the full sequence of $\{r_t^*, w_t^*\}_{t=0}^{\infty}$ (and $\pi_t^* = 0, \forall t$)
- Sufficient for the household to know the full path of **aggregate** capital stock $\{K_t^*\}_{t=0}^{\infty}$ and the **time-invariant mappings** from K_t^* to prices r_t^* and w_t^*, $\forall t$, such that

$$r_t^* = r(K_t^*), \quad w_t^* = w(K_t^*)$$

- In equilibrium, firm optimization requires these two mapping to satisfy

$$r(K_t^*) = F_1(K_t^*, 1), \quad w(K_t^*) = F_2(K_t^*, 1)$$

when labor markets clear and when K_t^* is optimal for the representative firm

More general case

- So household state variables should include

$$\underbrace{k}_{\text{individual state}} \quad , \quad \underbrace{K}_{\textbf{aggregate state}}$$

- Household control variable remains k'
- State variables next period: k', K', and a household needs to know K'
- Let the law of motion of aggregate capital be

$$K' = G(K)$$

- We can attempt to write the household's Bellman equation as

$$V(k, K) = \max_{k' \in [0, (r(K)+1-\delta)k + w(K)]} \{u((r(K) + 1 - \delta)k + w(K) - k') + \beta V(k', G(K))\}$$

with associated policy function $k' = g(k, K)$

Equilibrium definition

Definition 5

A **recursive competitive equilibrium** consists of household value function $V : \mathbb{R}_+^2 \to \mathbb{R}$ and policy function $g : \mathbb{R}_+^2 \to \mathbb{R}_+$, **pricing functions** $r, w : \mathbb{R}_+ \to \mathbb{R}_+$, and the aggregate law of motion $G : \mathbb{R}_+ \to \mathbb{R}_+$ such that

- (household optimization) given r, w and G, the solution to the household's Bellman equation is V with an associated policy g;
- (firm optimization) r, w satisfy $r(K) = F_1(K, 1)$ and $w(K) = F_2(K, 1)$, $\forall K$;
- (**consistency**) individual decision is consistent with aggregate law of motion

$$g(K, K) = G(K), \ \forall K.$$

Straightforward to verify that market clearing conditions hold.

"Big K, little k" trick and representative household

- The **consistency** condition once again requires that the **perceived** aggregate law of motion is induced by and compatible with individual decisions:

$$k' = g(k, K) = G(K) = K' \quad \text{whenever } k = K$$

- Why distinguishing k and K at all when $k = K$ in equilibrium with a representative household?
- Perfect competition! This is how we capture price-taking behavior.
  - taking prices as given $\iff$ taking aggregate state as given

To see it more clearly...

Equilibrium characterization

- FOC of household problem for $k' = g(k, K)$ (ignoring constraints)

$$u'((r(K) + 1 - \delta)k + w(K) - k') = \beta V_1(k', G(K))$$

- Envelope condition at k with $k' = g(k, K)$

$$V_1(k, K) = u'((r(K) + 1 - \delta)k + w(K) - k')(r(K) + 1 - \delta)$$

moving one period ahead with $K' = G(K)$ and $k'' = g(k', K')$

$$V_1(k', K') = u'((r(K') + 1 - \delta)k' + w(K') - k'')(r(K') + 1 - \delta)$$

- Observe: without distinguishing k' and K', the household would perceive that their choice of k' would affect prices $r(k')$ and $w(k')$

Equilibrium characterization

- The Euler equation with $k' = g(k, K)$, $K' = G(K)$, and $k'' = g(k', K') = g(g(k, K), G(K))$

$$u'((r(K) + 1 - \delta)k + w(K) - k') = \beta u'((r(K') + 1 - \delta)k' + w(K') - k'')(r(K') + 1 - \delta)$$

- Evaluated at the equilibrium, it coincides with the planner's Euler equation

Taking stock

What can we say about the RCE?

- An equilibrium concept to accommodate recursive individual decision problems
- Most often used in practice, especially with uncertainty and heterogenous agents
- Can be characterized and solved/computed
- Algorithms typically involve nested loops:
  - an outer loop deals with the aggregate law of motion, consistency, market clearing
  - an inner loop solves individual decisions via DP

Calibration and Quantitative Analysis

Functional-form assumptions

- CRRA instantaneous utility function

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma},\ \sigma > 0$$

- CRS Cobb-Douglas aggregate production function

$$F(K, L) = K^{\alpha} L^{1-\alpha},\ \alpha \in (0, 1)$$

Calibration (with growth)

If we consider population growth and technological progress in labor-augmenting productivity

$$L_t = (1+n)^t, \quad A_t = (1+\gamma)^t$$

and read directly from data: $n = 0.01$, and $\gamma = 0.02$. Remaining parameters

$$\beta, \sigma, \alpha, \delta$$

- Labor share (or equivalently, capital share): $1 - \alpha = w_t L_t / Y_t \implies \alpha = 1/3$
- Investment rate: recall that along the BGP, labor-adjusted investment satisfies $\widetilde{i}_{BGP} = (n + \gamma + \delta + n\gamma)\widetilde{k}_{BGP}$, so we match to

$$\frac{I_t}{K_t} = \frac{I_t/Y_t}{K_t/Y_t} \approx \frac{0.18}{3} = n + \gamma + \delta + n\gamma \implies \delta = 0.03$$

where we read from the data investment-output ratio and capital-output ratio

Calibration (with growth)

- The Euler equation along the BGP implies

$$r_{BGP} = f'(\widetilde{k}_{BGP}) = \frac{(1+n)(1+\gamma)^{\sigma}}{\beta} - 1 + \delta$$

$$r_{BGP} = \frac{\alpha Y_t}{K_t} = \frac{1/3}{3} = \frac{1.01 \times 1.02^{\sigma}}{\beta} - 1 + 0.03$$

  1 equation and 2 unknowns.

- We can't pin down σ using this model, so we typically get it from somewhere else. The usual choice is $\sigma = 1$ (log-utility) or $\sigma = 2$. If we pick $\sigma = 1$, then $\beta = 0.953$.

  See AKMM Ch.4.1 and Krueger Ch.3.4 for more discussions.

Calibration (no growth)

If we ignore sources of growth and set $n = \gamma = 0$:

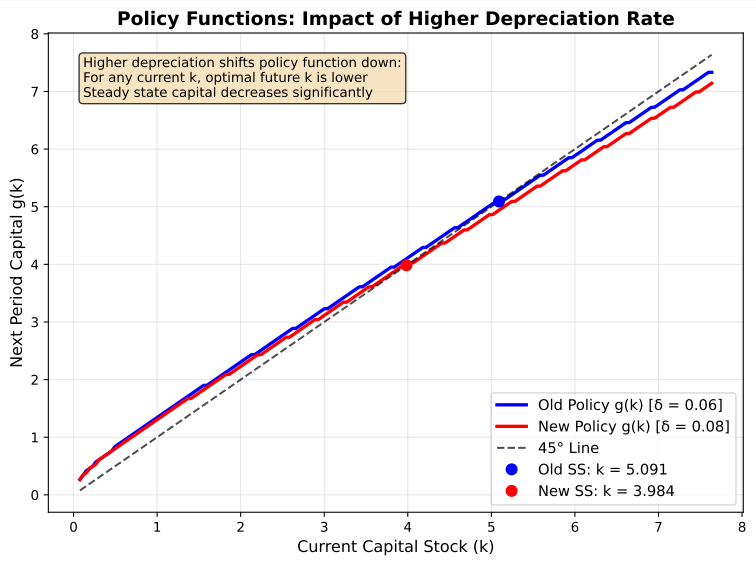
- Labor or capital share $\implies \alpha = 1/3$
- Investment rate $\implies \delta = 0.06$
- Externally pick $\sigma = 1$
- Capital-output ratio $\implies \beta = 0.95$

A quantitative exercise

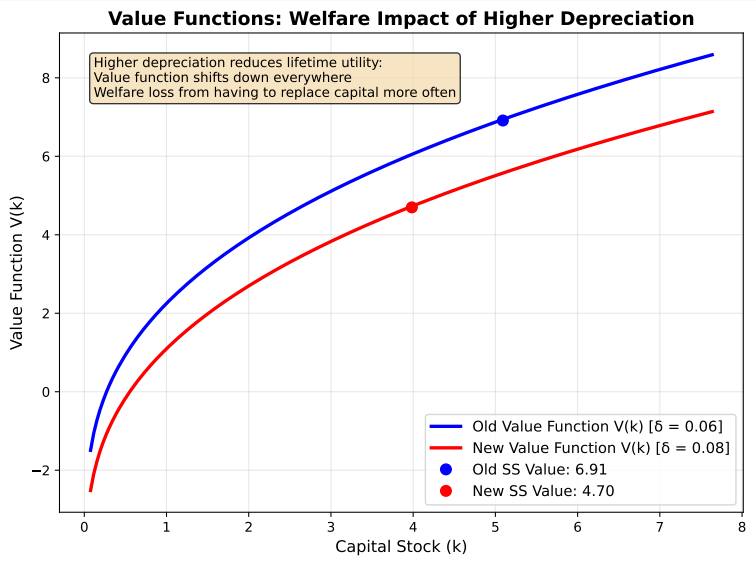
Suppose the economy is at its steady state equilibrium determined by the parameter values on the last page. What if the capital depreciation rate permanently increases to $\delta = 0.08$?

- This illustrates the dynamic general equilibrium adjustment
- We'll trace how the economy transitions from old to new steady state
- All prices and quantities adjust simultaneously along the transition path

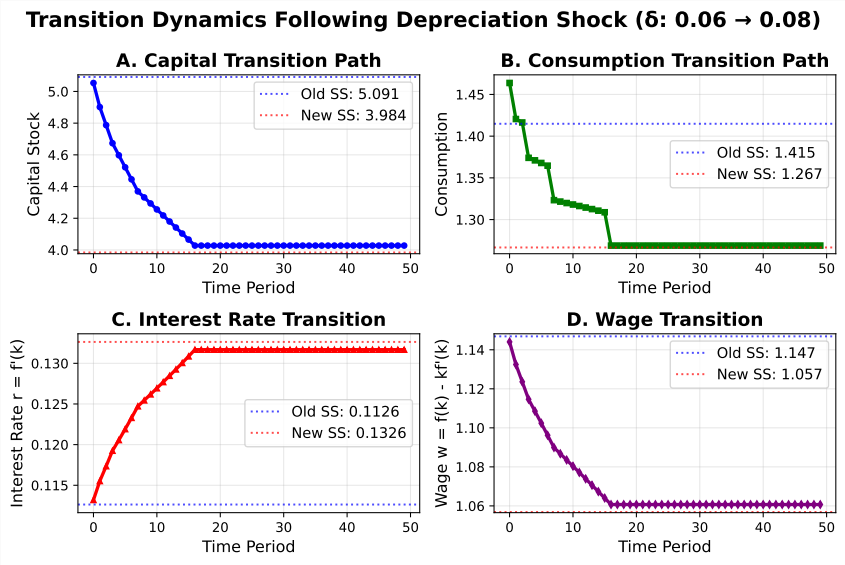
Policy functions



Value functions



Transition dynamics



The Two Welfare Theorems

The two welfare theorems

We state without proving the two welfare theorems in infinite-horizon models:

The First Welfare Theorem. If preferences satisfy local nonsatiation, then a competitive (Arrow-Debreu) equilibrium is Pareto optimal.

The Second Welfare Theorem. If consumption sets are convex, utility functions are continuous and strictly quasi-concave, and the aggregate production set has an interior point, then a Pareto-optimal allocation can be supported by a price system as a quasi-equilibrium, which is a competitive equilibrium if every consumption set contains a cheaper alternative given this price system.

For a formal proof/discussion, see Krueger Ch.7 or SLP Ch.15. (The proof of the first welfare theorem in AKMM Ch.6 does not directly apply to neoclassical growth models.)

Readings

- AKMM: Ch.5
- Krueger: Ch.3.3-3.4