ECON 720: Advanced Macroeconomic Theory I

Dynamic Optimization and Optimal Growth

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Neoclassical growth model

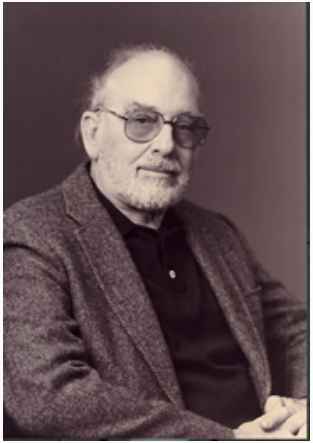
- Arguably the most important workhorse model in modern macro
  - it is used to study both long-run growth and short-run business cycle fluctuations
  - can be generalized to accommodate richer heterogeneity in agents
- We start from the **simplest version** of the model: deterministic (no uncertainty), inelastic labor supply (no disutility of labor), no productivity/population growth
- Essentially, it **endogenizes the saving decision** in the Solow model
  - explicit modeling of household optimization problem $\rightarrow$ saving responds to incentives
  - welfare and policy discussions made possible

Neoclassical growth model

The neoclassical growth model is often called the Ramsey-Cass-Koopmans model.



Frank P. Ramsey



David Cass



Tjalling Koopmans

Basic setup: environment

Very similar to the basic Solow model, except for household decisions.

- Discrete time, infinite horizon, $t = 0, 1, 2, \ldots$ (we'll start from the finite-horizon case)
- Agents: a representative household (or a unit measure of identical households) and a representative firm (or a unit measure of identical firms)
- Goods/commodities traded in each period: labor service, capital service, a final output good that can be either consumed or invested
- Households work, rent out capital, consume, and make capital investment, $\forall t$
- The representative firm hires labor and rents capital, and uses them as inputs to produce the consumption good according to a given technology, $\forall t$

Basic setup: technology

Production technology is represented by a time invariant **aggregate production function**

$$Y = F(K, L)$$

with the following **assumptions:**

- $F(K, L)$ exhibits CRS and is strictly increasing and concave in K and L
- $F(0, L) = F(K, 0) = 0$, and (Inada conditions) $\lim_{K \to 0} F_1(K, L) = \infty$, $\lim_{K \to \infty} F_1(K, L) = 0$

Basic setup: households

A unit measure of **identical** households, infinitely lived, with preferences over life-time consumption plans represented by a **time separable life-time utility function**

$$U(\underbrace{\{c_t\}_{t=0}^{\infty}}_{\text{life-time consumption}}) = \sum_{t=0}^{\infty} \overbrace{\beta^t}^{\text{discount future}} \times u(\underbrace{c_t}_{\text{time-}t\text{ consumption}}),$$

where

- $u(c_t)$ is the instantaneous/per-period utility function at each time t
  assumptions: u is at least twice differentiable, $u' > 0$ (strictly increasing), $u'' < 0$ (strictly concave), and satisfies the Inada conditions: $\lim_{c\to 0} u'(c) = \infty$ and $\lim_{c\to\infty} u'(c) = 0$ (these conditions help ensure unique interior solution to the household problem)

- $\beta \in (0, 1)$ is the discount factor

Basic setup: households

- Endowment: each household is endowed with $\overline{k}_0 > 0$ units of capital at $t = 0$ and 1 unit of productive time in each period t
- Given the capital stock k_t at t, if a household decides to invest i_t in period t, the capital available *next period* is

$$k_{t+1} = (1 - \delta)k_t + i_t,$$

  where $\delta \in [0, 1]$ is the capital depreciation rate, and we require $k_{t+1} \geq 0$, $\forall t$, but investment i_t may turn negative
- Agents have **perfect foresight**

Basic setup: equilibrium concept

- We start from *optimal growth*, that is, a **social planner's problem**
  - No need to specify market structure such as perfect competition
- We then move on to *competitive growth* (next deck of slides), where we do need to specify market structure and define a **competitive equilibrium**

Overview

Plan: **endogenize the saving rate** in **optimal growth**

- Set up the planner's baseline (∞-horizon) problem
- Build the technique up by *horizon*, solving the same problem four times:
  - 2 periods, then T periods: Lagrangian and KKT $\rightarrow$ **Euler equations**
  - the same finite problem *again*, by **backward induction** $\rightarrow$ value and policy functions
  - ∞ horizon, **sequential problem** (SP) $\rightarrow$ Euler equations + transversality condition
  - ∞ horizon, **recursive formulation** (FE) $\rightarrow$ the Bellman equation and more
- Then: steady state, the modified golden rule, convergence, the Solow comparison, and long-run growth

Optimal Growth

A social planner

- Imagine a benevolent social planner that chooses an allocation to maximize social welfare subject to technology and resource constraints
- Decision is made in period $t = 0$
- Welfare measure: household *life-time* utility $\sum_{t=0}^{\infty} \beta^t u(c_t)$
- Allocation: a sequence of $\{c_t, i_t, k_t, l_t\}_{t=0}^{\infty}$
- Why do we start from a planner's problem?
  - much easier to solve
  - efficient or first-best benchmark

Feasibility and Pareto efficiency

Before we set up the planner's problem, let's define the following:

Definition 1 (Feasibility)
An allocation $\{c_t, i_t, k_t, l_t\}_{t=0}^{\infty}$ is **feasible** if for all $t \geq 0$:

$$\begin{aligned} c_t + i_t \leq & F(k_t, l_t), \\ k_{t+1} = & (1-\delta)k_t + i_t, \end{aligned}$$

and $c_t, k_t \geq 0$ with $k_0 \leq \overline{k}_0$, and $l_t \in [0, 1]$.

Definition 2 (Pareto efficiency)
An allocation $\{c_t, i_t, k_t, l_t\}_{t=0}^{\infty}$ is **Pareto efficient** if it is feasible and there is no other feasible allocation $\{\widehat{c}_t, \widehat{i}_t, \widehat{k}_t, \widehat{l}_t\}_{t=0}^{\infty}$ such that (identical) households are strictly better off, or

$$\sum_{t=0}^{\infty} \beta^t u(\widehat{c}_t) > \sum_{t=0}^{\infty} \beta^t u(c_t).$$

The planner's problem

The social planner's problem at $t = 0$ is to find the *Pareto efficient* allocation:

$$
\begin{aligned}
W(\overline{k}_0) = \max_{\{c_t, i_t, k_t, l_t\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t u(c_t) \\
\text{subject to } c_t + i_t \leq & F(k_t, l_t), \ \forall t, \\
k_{t+1} = & (1-\delta)k_t + i_t, \ \forall t, \\
c_t \geq & 0, \ \forall t, \\
k_t \geq & 0, \ \forall t, \\
l_t \in & [0, 1], \ \forall t, \\
\text{and } k_0 \leq & \overline{k}_0.
\end{aligned}
$$

$W(\overline{k}_0)$ is the maximal welfare this economy can achieve when the initial endowment of capital stock is $\overline{k}_0$. Its only input argument is $\overline{k}_0$ because everything else of the optimal allocation is endogenously chosen. The optimized W as a function of $\overline{k}_0$ is usually called the **value function**. We'll come back to this later.

The planner's problem

Some observations to simplify the planner's problem:

- the planner will use up capital endowment $k_0 = \overline{k}_0$ and labor endowment $l_t = 1$, $\forall t$
- Inada condition $\lim_{c \to 0} u'(c) = \infty$ ensures that the planner will never set $c_t = 0$
- we can combine the resource constraint and capital accumulation to eliminate i_t

The planner's problem can be rewritten as

$$\begin{aligned} W(\overline{k}_0) = \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{subject to } c_t + k_{t+1} \leq & F(k_t, 1) + (1-\delta)k_t, \ \forall t, \\ k_{t+1} \geq & 0, \ \forall t, \\ \text{with } k_0 = & \overline{k}_0. \end{aligned}$$

To ease notation, we let $f(k_t) = F(k_t, 1)$ and sometimes use $\widetilde{f}(k_t) = F(k_t, 1) + (1-\delta)k_t$.

The planner's problem

- Goal: to find the optimal sequence $\{c_t, k_{t+1}\}_{t=0}^{\infty}$
- How do we solve this infinite-dimensional optimization problem with an infinite number of constraints?!
  - Dynamic programming (we'll be more specific about what this is)
  - Analytical solution in very rare special cases; a computer is usually needed
- We start from finite-period (hence finite-dimensional) optimization problem and work towards the infinite-period case
- (Yes, we must go through all these troubles to endogenize the saving rate/investment rule in the Solow model!)
- (But! Worth it. Once we get the basic model right, we can add all kinds of bells and whistles to address issues like optimal taxation, welfare cost of business cycles, etc.)

2-Period Case

2-period case: setup

- Same model setup except there are only 2 periods, $t = 0, 1$.
- Planner's problem at $t = 0$ can be written as

$$\begin{aligned} W_1(\overline{k}_0) = \max_{c_0, c_1, k_1, k_2} & \left\{ u(c_0) + \beta u(c_1) \right\} \\ \text{subject to} \quad c_0 + k_1 \leq & f(\overline{k}_0) + (1-\delta)\overline{k}_0, \\ c_1 + k_2 \leq & f(k_1) + (1-\delta)k_1, \\ k_1 \geq & 0, \\ k_2 \geq & 0. \end{aligned}$$

  Subscript "1" of $W_1(\overline{k}_0)$ means the horizon, i.e., there's 1 period left at/after $t = 0$. We ignore the constraints $c_0, c_1 \geq 0$ because we know they will never be binding.

- Our assumptions on u and F (hence f) ensure that (i) the objective function is concave and (ii) the choice set is convex, so the Karush-Kuhn-Tucker (KKT) conditions are both sufficient and necessary for optimal solution $c_0^*, c_1^*, k_1^*, k_2^*$.

2-period case: Lagrangian and KKT conditions

We write the **Lagrangian** as

$$\begin{aligned}&\mathcal{L}(c_0, c_1, k_1, k_2, \lambda_0, \lambda_1, \mu_0, \mu_1)\\ =&u(c_0) + \beta u(c_1) + \lambda_0[f(\overline{k}_0) + (1-\delta)\overline{k}_0 - k_1 - c_0] + \lambda_1[f(k_1) + (1-\delta)k_1 - k_2 - c_1]\\ &+ \mu_0 k_1 + \mu_1 k_2.\end{aligned}$$

KKT conditions include:

- the FOCs

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial c_0} =&u'(c_0^*) - \lambda_0^* = 0,\\ \frac{\partial \mathcal{L}}{\partial c_1} =&\beta u'(c_1^*) - \lambda_1^* = 0,\\ \frac{\partial \mathcal{L}}{\partial k_1} =&-\lambda_0^* + \lambda_1^*[f'(k_1^*) + 1 - \delta] + \mu_0^* = 0,\\ \frac{\partial \mathcal{L}}{\partial k_2} =&-\lambda_1^* + \mu_1^* = 0.\end{aligned}$$

- the complementary slackness conditions

$$\begin{aligned}\lambda_0^*[f(\overline{k}_0) + (1-\delta)\overline{k}_0 - k_1^* - c_0^*] =&0\\ \lambda_1^*[f(k_1^*) + (1-\delta)k_1^* - k_2^* - c_1^*] =&0\\ \mu_0^* k_1^* =&0\\ \mu_1^* k_2^* =&0\end{aligned}$$

- the constraints, and $\lambda_0^*, \lambda_1^*, \mu_0^*, \mu_1^* \geq 0$.

2-period case: a closer look at KKT conditions

Some observations:

- Inada condition $u'(0^+) = \infty$ already tells us $c_0^*, c_1^* > 0$, so $\lambda_0^*, \lambda_1^* > 0$:

  $$u'(c_0^*) = \lambda_0^*, \quad \beta u'(c_1^*) = \lambda_1^*,$$

  the household eats in each period and uses up all resources

- production at $t = 1$ must take place for $c_1^* > 0$ to be feasible, so $k_1^* > 0$ and $\mu_0^* = 0$

- $\lambda_1^* > 0$ implies $\mu_1^* > 0$ by the last FOC, so

  $$k_2^* = 0,$$

  i.e.,

  $$c_1^* = f(k_1^*) + (1 - \delta)k_1^*,$$

  the household simply eats everything if the world ends tomorrow

The Euler equation

Combining everything yields the **Euler equation**, a central element in macro:

$$\underbrace{u'(c_0^*)}_{\substack{\text{utility loss at } t=0 \text{ for} \\ \text{investing 1 unit more} \\ \text{capital for } t=1}} = \underbrace{\beta u'(c_1^*)}_{\substack{\text{discounted utility gain} \\ \text{for 1 unit more} \\ \text{consumption at } t=1}} \times \underbrace{[f'(k_1^*)+1-\delta]}_{\substack{\text{add'l output at } t=1 \text{ with} \\ \text{1 unit more capital}}}$$

Why so important?

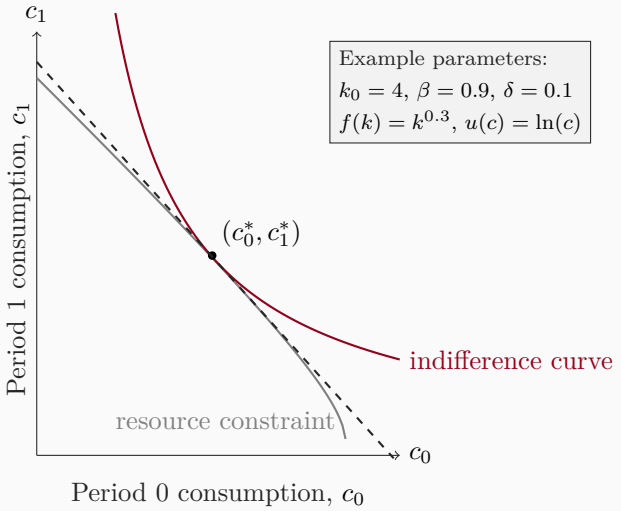
- saving/investment is a **dynamic decision**: capital tomorrow k_1 is determined today
- **intertemporal tradeoffs** of saving/investment decisions: MC today = MB tomorrow

In this 2-period case, solving the planner's problem boils down to solving for k_1^*:

$$u'(\underbrace{f(\overline{k}_0)+(1-\delta)\overline{k}_0-k_1^*}_{=c_0^*}) = \beta u'(\underbrace{f(k_1^*)+(1-\delta)k_1^*}_{=c_1^*})[f'(k_1^*)+1-\delta]$$

with $k_0=\overline{k}_0$ known and $k_2^*=0$.

The Euler equation illustrated



- indifference curve

$$W_1(k_0) = u(c_0) + \beta u(c_1)$$

- resource constraint

$$\begin{aligned} c_1 =& f(k_1) + (1-\delta)k_1 \\ =& f(f(k_0) + (1-\delta)k_0 - c_0) \\ &+ (1-\delta)[f(k_0) + (1-\delta)k_0 - c_0] \end{aligned}$$

- common tangent:
  the **Euler equation**

$$-\frac{dc_1}{dc_0} = \frac{u'(c_0^*)}{\beta u'(c_1^*)} = f'(k_1^*) + (1-\delta)$$

An analytical example

- Let $u(c) = \ln c$ and $f(k) = zk^\alpha$, $\alpha \in (0,1)$. Assume full depreciation $\delta = 1$, so $k_1 = i_0$.
- The Euler equation becomes

$$\frac{1}{c_0^*} = \beta \frac{1}{c_1^*} \alpha z k_1^{*\alpha-1} \quad \Longrightarrow \quad \frac{1}{z\bar{k}_0^\alpha - k_1^*} = \beta \frac{1}{z k_1^{*\alpha}} \alpha z k_1^{*\alpha-1},$$

and the solution is simply

$$k_1^* = \frac{\alpha\beta}{1+\alpha\beta} z\bar{k}_0^\alpha,$$

that is, the *optimal saving rate* at $t = 0$ is

$$s_0^* = \frac{i_0^*}{f(\bar{k}_0)} = \frac{k_1^*}{z\bar{k}_0^\alpha} = \frac{\alpha\beta}{1+\alpha\beta}.$$

The rest of the optimal allocation c_0^*, c_1^* follows. Note how s_0^* increases in β: save more if more patient.

Finite-Period Case

Finite-period case: setup

- Now we take one step further and consider the same model with a finite horizon $T < \infty$, $t = 0, 1, \ldots, T$
- Planner's problem at $t = 0$ can be written as

$$\begin{aligned} W_T(\overline{k}_0) = & \max_{\{c_t, k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(c_t) \\ \text{subject to } \quad & c_t + k_{t+1} \leq f(k_t) + (1-\delta)k_t, \ \forall t = 0, 1, \ldots, T, \\ & k_{t+1} \geq 0, \ \forall t = 0, 1, \ldots, T, \ k_0 = \overline{k}_0. \end{aligned}$$

  Subscript T of $W_T(\overline{k}_0)$ still means the horizon, i.e., there are T periods left at/after $t = 0$. We again ignore the constraints $c_t \geq 0$ because we know they will never be binding.

- Our assumptions on u and F (hence f) ensure that the KKT conditions remain both sufficient and necessary for optimal solution.

Finite-period case: Lagrangian and KKT conditions

We write the **Lagrangian** as

$$\mathcal{L}\left(\{c_t, k_{t+1}\}_{t=0}^{T}, \{\lambda_t, \mu_t\}_{t=0}^{T}\right) = \sum_{t=0}^{T}\Big(\beta^t u(c_t) + \lambda_t[f(k_t) + (1-\delta)k_t - k_{t+1} - c_t] + \mu_t k_{t+1}\Big).$$

KKT conditions include:

- the FOCs

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t} =& \beta^t u'(c_t^*) - \lambda_t^* = 0, \ \forall t = 0, 1, \dots T, \\
\frac{\partial \mathcal{L}}{\partial k_{t+1}} =& -\lambda_t^* + \lambda_{t+1}^*[f'(k_{t+1}^*) + 1 - \delta] + \mu_t^* = 0, \\
& \forall t = 0, 1, \dots T-1, \\
\frac{\partial \mathcal{L}}{\partial k_{T+1}} =& -\lambda_T^* + \mu_T^* = 0.
\end{aligned}$$

- the complementary slackness conditions for all $t = 0, 1, \dots T$:

$$\begin{aligned}
\lambda_t^*[f(k_t^*) + (1-\delta)k_t^* - k_{t+1}^* - c_t^*] =& 0, \\
\mu_t^* k_{t+1}^* =& 0.
\end{aligned}$$

- the constraints, and $\lambda_t^*, \mu_t^* \geq 0, \ \forall t = 0, 1, \dots T$.

Finite-period case: a closer look at KKT conditions

Some observations:

- We know $c_t^* > 0$, so $\lambda_t^* > 0$:

$$\beta^t u'(c_t^*) = \underbrace{\lambda_t^*}_{\substack{\text{discounted utility gain with 1 unit} \\ \text{increase in time-}t\text{ resource}}}, \ \forall t = 0, 1, \dots T$$

  the household eats in every period and uses up all resources

- production at t must take place for $c_t > 0$ to be feasible, so $k_t^* > 0$ and $\mu_t^* = 0$

- $\lambda_T^* > 0$ implies $\mu_T^* > 0$ by the last FOC, so

$$k_{T+1}^* = 0,$$

  i.e.,

$$c_T^* = f(k_T^*) + (1 - \delta)k_T^*,$$

  the household simply eats everything if the world ends tomorrow

The Euler equations

Combining everything yields the **Euler equation** system of T equations

$$\underbrace{u'(c_t^*)}_{\substack{\text{utility loss at } t \text{ for} \\ \text{investing 1 unit more} \\ \text{capital for } t+1}} = \underbrace{\beta u'(c_{t+1}^*)}_{\substack{\text{discounted utility gain} \\ \text{for 1 unit more} \\ \text{consumption at } t+1}} \times \underbrace{[f'(k_{t+1}^*) + 1 - \delta]}_{\substack{\text{add'l output at } t+1 \text{ with} \\ \text{1 unit more capital}}}$$

for all $t = 0, 1, \ldots, T-1$.

Same idea as in the 2-period case:

- **dynamic** investment decision: capital stock k_{t+1} at $t+1$ is decided one-period ahead in t (by choosing i_t)
- **intertemporal** tradeoff: MC at t = MB at $t+1$

The Euler equations

Solving the planner's problem boils down to solving the system of T **Euler equations for** T **unknowns** $\{k_{t+1}\}_{t=0}^{T-1}$:

$$u'(\underbrace{f(k_t^*) + (1-\delta)k_t^* - k_{t+1}^*}_{=c_t^*}) = \beta u'(\underbrace{f(k_{t+1}^*) + (1-\delta)k_{t+1}^* - k_{t+2}^*}_{=c_{t+1}^*})[f'(k_{t+1}^*) + 1 - \delta],$$

for all $t = 0, 1, \ldots, T-1$.

Recall: we know the **initial condition** $k_0^* = \overline{k}_0$ and the **terminal condition** $k_{T+1}^* = 0$.

We typically need a computer to solve this system numerically.

An analytical example

- Let $u(c) = \ln c$ and $f(k) = zk^\alpha$, $\alpha \in (0,1)$. Assume full depreciation $\delta = 1$.
- The system of Euler equations becomes

$$\frac{1}{c_t^*} = \beta \frac{1}{c_{t+1}^*} \alpha z k_{t+1}^{*\alpha-1} \qquad \Longrightarrow \qquad \frac{1}{z k_t^{*\alpha} - k_{t+1}^*} = \beta \frac{1}{z k_{t+1}^{*\alpha} - k_{t+2}^*} \alpha z k_{t+1}^{*\alpha-1},$$

  for all $t = 0, 1, \ldots, T-1$. We can solve it by brute force knowing $k_{T+1}^* = 0$.

- Alternatively, with $\delta = 1$, let the *optimal saving rate* at t be $s_t^* := \frac{i_t^*}{f(k_t^*)} = \frac{k_{t+1}^*}{f(k_t^*)}$, s.t. $s_T^* = 0$. It is easy to rewrite the Euler equations and solve it *backwards*:

$$\frac{s_t^*}{1 - s_t^*} = \frac{\alpha\beta}{1 - s_{t+1}^*} \implies s_t^* = \frac{\alpha\beta}{1 + \alpha\beta - s_{t+1}^*} \implies s_t^* = \frac{\sum_{j=1}^{T-t} (\alpha\beta)^j}{\sum_{j=0}^{T-t} (\alpha\beta)^j} = \alpha\beta \frac{1 - (\alpha\beta)^{T-t}}{1 - (\alpha\beta)^{T-t+1}},$$

  $\forall t = 0, 1, \ldots, T-1$. The rest of the optimal allocation follows. Note how s_t^* increases in β.

A Different Angle: Backward Induction

2-period case revisited

Knowing resource constraints will be binding, we can rewrite the planner's problem in the 2-period case as

$$
\begin{aligned}
W_1(\overline{k}_0) &= \max_{\substack{k_1\in[0,f(\overline{k}_0)+(1-\delta)\overline{k}_0]\\ k_2\in[0,f(k_1)+(1-\delta)k_1]}} \Big\{u(f(\overline{k}_0)+(1-\delta)\overline{k}_0-k_1)+\beta u(f(k_1)+(1-\delta)k_1-k_2)\Big\} \\
&= \max_{k_1\in[0,f(\overline{k}_0)+(1-\delta)\overline{k}_0]} \Big\{u(f(\overline{k}_0)+(1-\delta)\overline{k}_0-k_1) \\
&\qquad\qquad + \beta\times \underbrace{\max_{k_2\in[0,f(k_1)+(1-\delta)k_1]} \big\{u(f(k_1)+(1-\delta)k_1-k_2)\big\}}_{=W_0(k_1)} \Big\} \\
&= \max_{k_1\in[0,f(\overline{k}_0)+(1-\delta)\overline{k}_0]} \Big\{u(f(\overline{k}_0)+(1-\delta)\overline{k}_0-k_1)+\beta W_0(k_1)\Big\}
\end{aligned}
$$

What does this "nested" optimization problem mean? **Anticipating own optimal behavior at $t=1$, the planner/decision-maker chooses what's best at $t=0$.**

2-period case revisited

- So what's the planner's optimal behavior at $t = 1$, after which the world ends?
- Simple: whatever capital stock $k_1 > 0$ is when entering $t = 1$, the planner will always choose $k_2^* = 0$

$$\begin{aligned} W_0(k_1) = &\max_{k_2 \in [0, f(k_1)+(1-\delta)k_1]} \left\{ u(f(k_1) + (1-\delta)k_1 - k_2) \right\} \\ = &u(f(k_1) + (1-\delta)k_1) \end{aligned}$$

  Note the maximal welfare $W_0(k_1)$ is a function of k_1.

- Now we can go back to $t = 0$ and solve the problem at $t = 0$, knowing the planner will optimize at $t = 1$ and W_0 depends on the choice of k_1:

$$W_1(\overline{k}_0) = \max_{k_1 \in [0, f(\overline{k}_0)+(1-\delta)\overline{k}_0]} \left\{ u(f(\overline{k}_0) + (1-\delta)\overline{k}_0 - k_1) + \beta W_0(k_1) \right\}$$

2-period case revisited

- Optimization at $t = 0$ requires the FOC to hold (we ignore the constraints on k_1 knowing they won't be binding)

$$\begin{aligned} u'(f(\overline{k}_0) + (1-\delta)\overline{k}_0 - k_1^*) =& \beta W_0'(k_1^*) \\ =& \beta u'(f(k_1^*) + (1-\delta)k_1^*)[f'(k_1^*) + 1 - \delta] \end{aligned}$$

  which is just the Euler equation.

- This approach is **backward induction** as we start from the end period $t = 1$ and go backward in time to deal with the problem in $t = 0$

- To illustrate it more, let's **generalize it to any finite-horizon case**.

Another look at the finite-horizon case: period T

- Finite horizon means there is a last period T, after which the world ends. So whatever the capital stock k_T is when entering period T, the planner's problem at T is a simple static optimization problem:

$$W_0(k_T) = \max_{k_{T+1} \in [0, f(k_T)+(1-\delta)k_T]} u(f(k_T) + (1-\delta)k_T - k_{T+1})$$

- That is, the optimal capital stock is a trivial function of k_T:

$$k^*_{T+1} = g_0(k_T) = 0,$$

such that

$$W_0(k_T) = u(f(k_T) + (1-\delta)k_T).$$

- Observe that the period-T maximal welfare $W_0(k_T)$ strictly increases in the available capital stock k_T:

$$W_0'(k_T) = u'(f(k_T) + (1-\delta)k_T)[f'(k_T) + 1 - \delta] > 0.$$

Another look at the finite-horizon case: period $T-1$

- Going back to period $T-1$, the planner anticipates what they'll do in period T to achieve maximal welfare $W_0(k_T)$. If the available capital stock is k_{T-1} entering $T-1$, the planner's problem at t can be written as:

$$W_1(k_{T-1}) = \max_{k_T \in [0, f(k_{T-1})+(1-\delta)k_{T-1}]} \left\{ u(f(k_{T-1}) + (1-\delta)k_{T-1} - k_T) + \beta W_0(k_T) \right\}$$

- Observe that optimization requires FOC (we ignore the constraints on k_T knowing they won't be binding)

$$\begin{aligned} u'(f(k_{T-1}) + (1-\delta)k_{T-1} - k_T^*) =& \beta W_0'(k_T^*) \\ =& \beta u'(f(k_T^*) + (1-\delta)k_T^*)[f'(k_T^*) + 1 - \delta] \end{aligned}$$

where $W_0'(k_T)$ is obtained in the previous step. This is the **Euler equation** again.

Another look at the finite-horizon case: period $T-1$

- In other words, **if k_{T-1} is given, we know k_T^*, so the optimal k_T^* is a function of k_{T-1}**:

$$k_T^* = g_1(k_{T-1})$$

at which

$$W_1(k_{T-1}) = u(f(k_{T-1}) + (1-\delta)k_{T-1} - g_1(k_{T-1})) + \beta W_0(g_1(k_{T-1}))$$

- We can similarly look for $W_1'(k_{T-1})\ldots$

Another look at the finite-horizon case: period $T-1$

$$
\begin{aligned}
W_1'(k_{T-1}) =& \underbrace{u'(f(k_{T-1})+(1-\delta)k_{T-1}-g_1(k_{T-1}))[f'(k_{T-1})+1-\delta]}_{\text{direct effect}} \\
& \underbrace{-u'(f(k_{T-1})+(1-\delta)k_{T-1}-g_1(k_{T-1}))g_1'(k_{T-1})+\beta W_0'(g_1(k_{T-1}))g_1'(k_{T-1})}_{\text{indirect effect of } k_{T-1} \text{ on } W_1 \text{ through optimal } k_T^* = g_1(k_{T-1})} \\
=& u'(f(k_{T-1})+(1-\delta)k_{T-1}-g_1(k_{T-1}))[f'(k_{T-1})+1-\delta] \\
& \Big[\underbrace{-u'(f(k_{T-1})+(1-\delta)k_{T-1}-g_1(k_{T-1}))+\beta W_0'(g_1(k_{T-1}))}_{=0, \text{ by the FOC for } k_T^*}\Big] g_1'(k_{T-1}) \\
=& u'(f(k_{T-1})+(1-\delta)k_{T-1}-g_1(k_{T-1}))[f'(k_{T-1})+1-\delta] \\
& \qquad \text{(direct effect of } k_{T-1} \text{ only!)}
\end{aligned}
$$

This is an application of the **envelope theorem**. It essentially allows us to ignore any indirect effect through optimal choices.

Another look at the finite-horizon case: period t

- The same logic continues as we move further back in time:

$$W_{T-t}(k_t) = \max_{k_{t+1} \in [0, f(k_t)+(1-\delta)k_t]} \left\{ u(f(k_t) + (1-\delta)k_t - k_{t+1}) + \beta W_{T-t-1}(k_{t+1}) \right\}$$

  where **the optimal** k_{t+1}^* satisfies the FOC (constraints on k_{t+1} ignored)

$$\begin{aligned} u'(f(k_t) + (1-\delta)k_t - k_{t+1}^*) =& \beta W'_{T-t-1}(k_{t+1}^*) \\ =& \beta u'(f(k_{t+1}^*) + (1-\delta)k_{t+1}^* - g_{T-t-1}(k_{t+1}^*))[f'(k_{t+1}^*) + 1 - \delta], \end{aligned}$$

  where $k_{t+2}^* = g_{T-t-1}(k_{t+1}^*)$. This is the **Euler equation** linking periods t and $t+1$.

- So **the optimal** k_{t+1}^* is a function of k_t, s.t.

$$\begin{aligned} k_{t+1}^* =& g_{T-t}(k_t) \\ W_{T-t}(k_t) =& u(f(k_t) + (1-\delta)k_t - k_{t+1}^*) + \beta W_{T-t-1}(k_{t+1}^*) \end{aligned}$$

Another look at the finite-horizon case: period 0

- Eventually, back at time $t = 0$, the planner's problem can be written as

$$W_T(\overline{k}_0) = \max_{k_1 \in [0, f(\overline{k}_0)+(1-\delta)\overline{k}_0]} \left\{ u(f(\overline{k}_0) + (1-\delta)\overline{k}_0 - k_1) + \beta W_{T-1}(k_1) \right\}$$

with optimization condition (constraints on k_1 ignored)

$$\begin{aligned} u'(f(\overline{k}_0) + (1-\delta)\overline{k}_0 - k_1^*) =& \beta W'_{T-1}(k_1^*) \\ =& \beta u'(f(k_1^*) + (1-\delta)k_1^* - g_{T-1}(k_1^*))[f'(k_1^*) + 1 - \delta] \end{aligned}$$

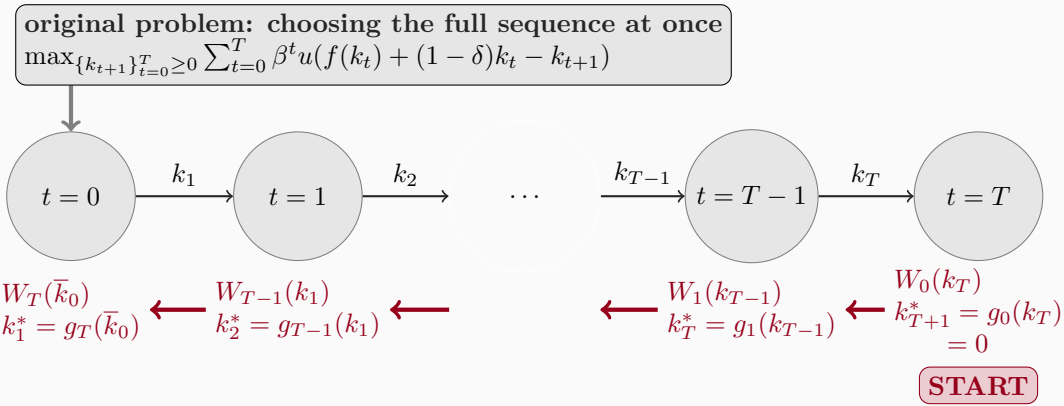
This is the **Euler equation** linking periods $t = 0$ and $t = 1$.

- So the optimal k_1^* can be viewed as a function of $k_0 = \overline{k}_0$:

$$\begin{aligned} k_1 =& g_T(k_0) \\ W_T(k_0) =& u(f(\overline{k}_0) + (1-\delta)\overline{k}_0 - k_1^*) + \beta W_{T-1}(k_1^*) \end{aligned}$$

Another look at the finite-horizon case

Taking stock: original problem vs. "backward induction" exercise



Backward induction: period-by-period decisions, working backward in time

Another look at the finite-horizon case

- Why did we solve the planner's problem in 2 different but ultimately equivalent ways?
  - one deals with the *sequential formulation* directly, where the objective function is $\sum_{t=0}^{T} \beta^t u(c_t)$
  - the other writes the same problem using value functions $W_{T-t}(k_t)$
- Because this is a prototype idea of **dynamic programming** in finite horizon!
- Note how we iteratively work with **value functions** $W_{T-t}(k_t)$ and associated **policy functions** $k_{t+1}^* = g_{T-t}(k_t)$ *backward* in time.
  - This backward recursion is the heart of dynamic programming with infinite horizon. We'll use this pattern repeatedly

Infinite Horizon: Sequence Problem

From finite horizon to infinite horizon

- Back to the original model setup with infinite horizon, i.e., $T \to \infty$
- It's tempting to think that we'd solve a (sequence of) T-period problem first (e.g., find $\{k_{t+1}^*\}_{t=0}^{T}$) and then send $T \to \infty$
  - it works in very rare cases (e.g., when we know k_{t+1}^* as an explicit function of t)
  - almost always infeasible in practice
- We'll need dynamic programming (DP) techniques
- **But why bother dealing with an infinite-horizon problem to begin with?**
  - T is the time a modeled economy/world ends, better treated as indefinite or unknown ($\beta < 1$ captures some flavor of world ending randomly)
  - but human lives are finite: sure, we deal with it in overlapping-generation (OLG) models
  - an infinite-horizon model can be a lot easier than a finite-period problem with large T

Roadmap for the rest of this deck

Everything from here on is one of six moves.

Economic question	**Mathematical object**	**Why *this* math**
How much should HH save?	max over infinite sequences (SP)	infinite horizon, dynamic decision
Intertemporal tradeoff	the **Euler equation**	gives economic intuition, not a solution
Limiting behavior at $t = \infty$	the **TVC**	replaces terminal condition
Same problem every period	the **Bellman equation** (FE)	*one function* instead of an infinite sequence
(SP) $\iff$ (FE)?	a few theorems	existence, uniqueness, *and* $V = W$
How do we solve it?	**value function iteration**	the algorithm implied by CMT

Simplifying the planner's problem

- For simplicity, let's write the planner's problem as

$$W(\overline{k}_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) + (1-\delta)k_t - k_{t+1}) \qquad \text{(SP)}$$

$$\text{subject to} \quad k_{t+1} \in [0, f(k_t) + (1-\delta)k_t], \forall t, \quad \text{and } k_0 = \overline{k}_0.$$

We can do this because we know the resource constraint will always be binding, i.e., nothing will be wasted, so we replace $c_t = f(k_t) + (1-\delta)k_t - k_{t+1}$

- We refer to a problem in this form as a **sequence problem (SP)** (because we are choosing an optimal sequence of $\{k_{t+1}\}_{t=0}^{\infty}$)

A technical side note before we move on...

- Technically, we should replace "**max**" with "**sup**" in (SP) because the infinite series $\sum_{t=0}^{\infty} \beta^t u(c_t)$ may go to infinity
- We make additional assumptions to make sure it does not happen:
  - either we directly impose $\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t u(c_t)$ exists and is finite for any feasible allocation given any finite $\overline{k}_0 > 0$
  - or, we simply require u to be bounded (we stick to this one)

Further simplifying the planner's problem

- We also know that constraints on k_{t+1} are never binding in any optimal allocation, $\forall t$
  - if $k_{t+1} = 0$: households eat nothing at $t+1$, i.e., $c_{t+1} = 0$, Inada condition $u'(0^+) = \infty$ says planner can do infinitely better by setting $c_{t+1} > 0$
  - if $k_{t+1} = f(k_t) + (1-\delta)k_t$: households eat nothing at t, i.e., $c_t = 0$, same argument
- So we can safely ignore the constraints on k_{t+1} for now and focus on interior solutions

The Euler equations

Next: the Euler equations

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Necessity of the Euler equations

- Our earlier assumptions ensure that FOCs, i.e., the **Euler equations** remain *necessary* conditions for any optimal sequence $\{k_{t+1}^*\}_{t=0}^{\infty}$
- To see why: given an optimal sequence of $\{k_{t+1}^*\}_{t=0}^{\infty}$, any element k_{t+1}^* must solve the local, 2-period problem:

$$\max_{k_{t+1}} u(f(k_t^*) + (1-\delta)k_t^* - k_{t+1}) + \beta u(f(k_{t+1}) + (1-\delta)k_{t+1} - k_{t+2}^*)$$

subject to the never-binding constraints $k_{t+1} \in [0, f(k_t^*) + (1-\delta)k_t^*]$ and $k_{t+2}^* \in [0, f(k_{t+1}) + (1-\delta)k_{t+1}]$. Its FOC is the Euler equation linking t and $t+1$.

The Euler equations

The **Euler equations** look the same, except we now have an **infinite number** of them:

$$u'(\underbrace{f(k_t^*) + (1-\delta)k_t^* - k_{t+1}^*}_{=c_t^*}) = \beta u'(\underbrace{f(k_{t+1}^*) + (1-\delta)k_{t+1}^* - k_{t+2}^*}_{=c_{t+1}^*})[f'(k_{t+1}^*) + 1 - \delta],$$

for all $t = 0, 1, 2, \ldots$

- **Problem?** We still have the initial condition $k_0^* = \overline{k}_0$, but we **no longer have the terminal condition** $k_{T+1}^* = 0$, because the world never ends.
- But we need a terminal condition to pin down an optimal solution.

The transversality condition

Next: the transversality condition

Economic question	**Mathematical object**	**Why *this* math**
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The transversality condition

- The **transversality condition (TVC)** serves the role of the missing terminal condition:

$$\textbf{TVC:}\quad \lim_{t\to\infty} \underbrace{\beta^t u'(\overbrace{f(k_t)+(1-\delta)k_t-k_{t+1}}^{=c_t})[f'(k_t)+1-\delta]}_{\text{value of one add'l unit of capital at } t \text{ in discounted utility}} \times \underbrace{k_t}_{\text{capital stock at } t} = 0$$

  Interpretation: present (period-0) value of period-t capital stock vanishes as $t \to \infty$.

- An alternative and equivalent way to write the TVC is

$$\lim_{t\to\infty} \beta^t u'(f(k_t)+(1-\delta)k_t-k_{t+1})k_{t+1} = 0$$

Where the TVC comes from

- TVC looks mysterious and scary. Where does it come from?
- Here's the "intuition" of it: recall the KKT conditions relevant for period-T decisions in the finite-horizon case:

$$\left.\begin{aligned} \beta^T u'(c_T) &= \lambda_T \\ \lambda_T &= \mu_T \\ \mu_T k_{T+1} &= 0 \end{aligned}\right\} \implies \lambda_T k_{T+1} = \beta^T u'(c_T) k_{T+1} = 0$$

- Sending $T \to \infty$ gives us the second TVC form.
- The equivalence between the two forms can be established using the Euler equation.
- If we rewrite per-period utility as $\widetilde{u}(k_t, k_{t+1}) = u(c_t)$, then TVC requires $\lim_{t\to\infty} \beta^t \widetilde{u}_1(k_t, k_{t+1}) k_t = 0$

Where the TVC comes from

Or, we can write the Lagrangian for the infinite-horizon problem *directly*, knowing constraints $k_{t+1} \geq 0,\ \forall t$, never bind:

$$\mathcal{L} = \sum_{t=0}^{\infty} \Big[\beta^t u(c_t) + \lambda_t \big(f(k_t) + (1-\delta)k_t - k_{t+1} - c_t\big)\Big]$$

- we collect terms by variables c_t and k_t and rewrite the Lagrangian

$$\begin{aligned}\mathcal{L} = &\sum_{t=0}^{\infty} \Big[\beta^t u(c_t) - \lambda_t c_t\Big] + \lambda_0[f(k_0) + (1-\delta)k_0] \\ &+ \sum_{t=1}^{\infty} \Big[\lambda_t[f(k_t) + (1-\delta)k_t] - \lambda_{t-1}k_t\Big] - \lim_{T\to\infty} \lambda_T k_{T+1}\end{aligned}$$

- FOCs: ∂c_t gives $\lambda_t = \beta^t u'(c_t)$; ∂k_t for $t \geq 1$ gives $\lambda_{t-1} = \lambda_t f'(k_t) + 1 - \delta$, i.e., the **Euler equation**

Where the TVC actually comes from

- Notice the boundary term: $-\lim_{T\to\infty} \lambda_T k_{T+1}$ now sits *inside the objective*. With $\lambda_T > 0$ and $k_{T+1} \geq 0$, the planner can only be optimizing if it vanishes:

$$\lim_{T\to\infty} \lambda_T k_{T+1} = \lim_{T\to\infty} \beta^T u'(c_T) k_{T+1} = 0 \tag{TVC}$$

  which is again the second TVC form.

- Caution:
  - "the planner wants it to vanish" is still a loose necessity argument
  - it requires $\mathcal{L}$'s components to be well-defined: ensured by $\beta \in (0,1)$ and a bounded u here, but KKT conditions don't always generalize to infinite dimensions!
  - see Kamihigashi (2001, Econometrica) for TVC's necessity in ∞-horizon models

Sufficiency of the Euler equations and TVC

Theorem 3 (Sufficiency of the Euler equations and TVC)
Suppose the earlier assumptions on u and F (hence f) hold, $\beta \in (0,1)$. Then an allocation $\{k_{t+1}\}_{t=0}^{\infty}$ that satisfies (i) the Euler equations and (ii) TVC solves the planner's sequence problem (SP), for a given $k_0 = \overline{k}_0 > 0$.

See SLP Theorem 4.15 for a more general statement and proof.

- It means that **IF** we find an allocation that satisfies the Euler equations and are able to verify that it meets TVC, we find our solution.
- **It does not address the practical "how to find such an allocation"** with very few exceptions
- The upcoming example is such an exception...

An analytical example

- Let $u(c) = \ln c$ and $f(k) = zk^\alpha$, $\alpha \in (0,1)$. Assume full depreciation $\delta = 1$. This is the same as the finite-horizon example.
- Solving the T-period problem tells us

$$s_t^* = \alpha\beta \frac{1-(\alpha\beta)^{T-t}}{1-(\alpha\beta)^{T-t+1}}, \quad \text{i.e., } k_{t+1}^* = \alpha\beta \frac{1-(\alpha\beta)^{T-t}}{1-(\alpha\beta)^{T-t+1}} z k_t^{*\alpha}, \ \forall t = 0, 1, \ldots, T-1.$$

- Sending $T \to \infty$, we obtain a candidate sequence $\{k_{t+1}^*\}_{t=0}^\infty$ s.t. $k_{t+1}^* = s^* z k_t^{*\alpha}$ where

$$s^* = \lim_{T\to\infty} \alpha\beta \frac{1-(\alpha\beta)^{T-t}}{1-(\alpha\beta)^{T-t+1}} = \alpha\beta.$$

  It is easy to verify that all Euler equations are satisfied. Note how s^* is constant (recall: Solow) and increases in β.

An analytical example

- Now we verify that this sequence satisfies TVC:

$$\lim_{t\to\infty} \beta^t u'(f(k_t^*) - k_{t+1}^*) f'(k_t^*) k_t^* = \lim_{t\to\infty} \beta^t \frac{\alpha f(k_t^*)}{f(k_t^*) - k_{t+1}^*} = \lim_{t\to\infty} \beta^t \frac{\alpha}{1-s^*} = 0$$

- So the candidate sequence generated by $k_{t+1}^* = \alpha\beta z k_t^\alpha$ from a given $\overline{k}_0$ is indeed a solution to (SP)
- The dynamics of this particular example coincides with its Solow counterpart
- But this equivalence is not guaranteed in more general cases! Why? Because the endogenous saving rate at t is typically a nontrivial function of k_t

Why did we have to check the TVC?

- Because it's possible to find a candidate sequence that
  - satisfies EE
  - doesn't satisfy TVC

  such a sequence is not optimal! See the example.
- EEs are a system of second-order difference equations: one initial condition ($k_0 = \overline{k_0}$) is not enough to ensure a unique solution sequence
- Think of TVC as a "selection rule"

Why did we have to check the TVC?

Same example ($u = \ln c$, $f(k) = zk^\alpha$, $\delta = 1$). What goes wrong if we only impose the Euler equations but ignore TVC?

- In saving rates $s_t := k_{t+1}/f(k_t)$, the Euler equation run *forward* in time is

$$s_{t+1} = 1 + \alpha\beta - \frac{\alpha\beta}{s_t}, \quad \text{with } \textbf{two} \text{ steady states } s = \alpha\beta \text{ and } s = 1.$$

- *Every* s_0 yields a path satisfying **all** Euler equations: they cannot pin down s_0
- Its slope $\alpha\beta/s^2$ makes $s = \alpha\beta$ **unstable** and $s = 1$ **stable** *going forward*:
  - $s_0 > \alpha\beta$: $s_t \to 1$, feasible forever, but consumption vanishes $\to$ TVC fails
  - $s_0 < \alpha\beta$: s_t falls and turns negative in finite time $\to$ infeasible
  - $s_0 = \alpha\beta$: the optimum we just found

Why did we have to check the TVC?

A numerical look, with $\alpha = 0.4$, $\beta = 0.75$ (so $\alpha\beta = 0.3$), starting from $s_0 = 0.5 > \alpha\beta$. Recall the TVC term is $\beta^t u'(c_t) f'(k_t) k_t = \beta^t \frac{\alpha}{1-s_t}$.

t	0	1	2	3	4	5	6	7
s_t	0.500	0.700	0.871	0.956	0.986	0.996	0.999	1.000
TVC term	0.80	1.00	1.75	3.81	9.11	22.46	55.91	139.59

- Every column satisfies the Euler equation linking it to the next. The economy keeps saving more and more, and the TVC term diverges instead of vanishing.
- **Takeaway:** the optimum is a *saddle* — backward-stable, forward-unstable.
  - in *finite* horizon, $k^*_{T+1} = 0$ let us build the path *backward*, along the stable direction: no TVC needed
  - in *infinite* horizon there is no last period to start from, so the TVC is what selects the one forward path that does not run away

Roadmap: where we are

The sequence problem is done. It gave us economics — but, except in knife-edge examples, no way to actually solve the model.

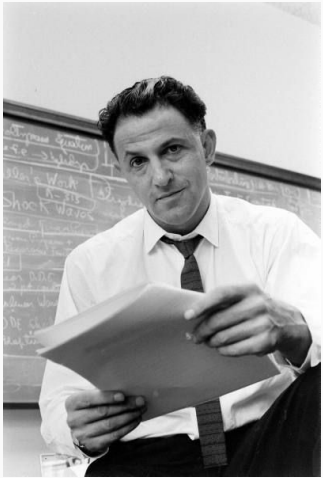
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Infinite Horizon: Dynamic Programming

Logic

- We've seen that the original sequence problem (SP) can be difficult to solve
- So we need to turn (SP) into something easier to solve
  - How? By exploiting the "stationarity" of the economic environment: there are always infinite number of periods left at any finite t, and the problem looks the same at any t except for some changing "state" of the world (e.g., k_t)
- Then we solve this easier problem
  - Dynamic programming: instead of looking for a *infinite sequence of numbers* (e.g., $\{k_{t+1}\}_{t=0}^{\infty}$), we look for *ONE function* (e.g., a time-invariant mapping from any k to next period's $k' = g(k)$)
- Lastly, we make sure the solution to the easier problem also solves the original (SP)

Principle of optimality



Richard E. Bellman

- Main idea of dynamic programming: break down a complex optimization problem (e.g., all infinite periods) into simpler subproblems (e.g., now and future)
- **Bellman's principle of optimality**: An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. (Bellman, 1957)

Basic idea

Let's illustrate the transformation as follows (a bit hand-wavy). Given a $k_0 > 0$:

$$
\begin{aligned}
W(k_0) &= \max_{\substack{\{k_{t+1}\}_{t=0}^{\infty},\ \text{s.t.} \\ k_{t+1}\in[0,f(k_t)+(1-\delta)k_t]}} \left\{ \sum_{t=0}^{\infty} \beta^t u(f(k_t)+(1-\delta)k_t - k_{t+1}) \right\} \\
&= \max_{\substack{\{k_{t+1}\}_{t=0}^{\infty},\ \text{s.t.} \\ k_{t+1}\in[0,f(k_t)+(1-\delta)k_t]}} \left\{ u(f(k_0)+(1-\delta)k_0 - k_1) + \beta \sum_{t=1}^{\infty} \beta^{t-1} u(f(k_t)+(1-\delta)k_t - k_{t+1}) \right\} \\
&= \max_{k_1\in[0,f(k_0)+(1-\delta)k_0]} \Bigg\{ u(f(k_0)+(1-\delta)k_0 - k_1) \\
&\qquad + \beta \times \underbrace{\max_{\substack{\{k_{t+2}\}_{t=0}^{\infty},\ \text{s.t.} \\ k_{t+2}\in[0,f(k_{t+1})+(1-\delta)k_{t+2}]}} \left\{ \sum_{t=0}^{\infty} \beta^t u(f(k_{t+1})+(1-\delta)k_{t+1} - k_{t+2}) \right\}}_{=W(k_1)}
\end{aligned}
$$

Basic idea

- The previous slide suggests

$$W(k_0) = \max_{k_1 \in [0, f(k_0)+(1-\delta)k_0]} \left\{ u(f(k_0) + (1-\delta)k_0 - k_1) + \beta W(k_1) \right\}$$

  Interpretation: anticipating optimal behavior from $t = 1$ onward and how the choice of k_1 affects the future, the planner/decision-maker optimally chooses k_1 at $t = 0$

- Stationarity: unlike the T-period case, $W(k)$ does not have a time-dependent subscript

- This is **one** (functional) **equation of one unknown** (function): W. The other function $k_1 = g(k_0)$ will follow.

- Questions:

  - When does this hand-wavy transformation actually hold?

  - How to solve **this** (easier) problem?

Recursive formulation of the planner's problem

Before we answer the two questions, let's take a closer look at the functional equation (FE):

$$V(k) = \max_{k' \in [0, f(k)+(1-\delta)k]} \left\{ u(f(k) + (1-\delta)k - k') + \beta V(k') \right\} \tag{FE}$$

- Notational convention: prime " $'$ " means next period's variable
  - k: capital stock this period
  - k': capital stock next period
- Which period is "this period"? It doesn't matter in a stationary model environment.
  - regardless of being in period 0, 100, or 1000, as long as the available capital stock k is the same, the planner/decision-maker solves the same infinite-horizon problem

Recursive formulation of the planner's problem

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$$V(k) = \max_{k' \in [0, f(k)+(1-\delta)k]} \left\{ u(f(k) + (1-\delta)k - k') + \beta V(k') \right\} \tag{FE}$$

- This is called the **recursive formulation** of an optimization problem: the same function V on both LHS and RHS of the maximization problem
- We usually call this functional equation the **Bellman equation**
- We write V instead of W: that (FE)'s solution equals (SP)'s value is a **theorem** to establish

Recursive formulation of the planner's problem

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- The Bellman equation is written in terms of the **state variable(s)** or simply the **state**, describing the current state of the world, that is, *everything that (i) the planner/decision-maker needs to know to make the optimal decision now and (ii) may change over time*
  - LHS, $V(k)$: this period's capital stock k is this period's state variable
  - RHS, $V(k')$: next period's capital stock k' is next period's state variable
- The variable the decision maker chooses is called the **control variable(s)**
  - in this case, the control variable now k' becomes the state variable next period

Recursive formulation of the planner's problem

Before we answer the two questions, let's take a closer look at the functional equation (FE):

$$V(k) = \max_{k' \in [0, f(k)+(1-\delta)k]} \left\{ u(f(k) + (1-\delta)k - k') + \beta V(k') \right\} \tag{FE}$$

- The solution to the Bellman equation in (FE) is the **value function** V, a function of the state:
  $V(k)$ is interpreted as the *maximal* discounted lifetime utility of the representative household from this period onward given *any* capital stock k at the beginning of the current period (instead of a particular level)

- Solving the Bellman equation will also give us the decision rule $k' = g(k)$, a mapping from the state to the optimal control. When g is a function (instead of a correspondence), it is called the **policy function**

  - i.e., we need to know the optimal choice of next period's capital stock given *any* current capital stock

Recursive formulation of the planner's problem

Before we answer the two questions, let's take a closer look at the functional equation (FE):

$$V(k) = \max_{k' \in [0, f(k)+(1-\delta)k]} \left\{ u(f(k) + (1-\delta)k - k') + \beta V(k') \right\} \tag{FE}$$

- The relationship between the value function V and its policy function g is

$$V(k) = u(f(k) + (1-\delta)k - g(k)) + \beta V(g(k))$$

- Once we know g, and if we can establish equivalence between (SP) and (FE), we can easily get the sequence $\{k^*_{t+1}\}_{t=0}^{\infty}$ as $k^*_{t+1} = g(k^*_t)$ starting from *any* k_0
  - In contrast, (SP)'s solution is only one such sequence starting from a particular $\overline{k}_0$

General forms of SP and FE

- A more general form of (SP) looks like this, where $X \subseteq \mathbb{R}^l$ is the set of all possible state variable values, $\Gamma : X \rightrightarrows X$ describes the feasibility constraints, $\mathcal{F} : \{(x, y) : x \in X, y \in \Gamma(x)\} \to \mathbb{R}$ is the per-period reward:

$$W(x_0) = \max_{\{x_{t+1}\}_{t=0}^{\infty}} \left\{ \sum_{t=0}^{\infty} \beta^t \mathcal{F}(x_t, x_{t+1}) \right\} \text{ s.t. } \quad x_{t+1} \in \Gamma(x_t), \forall t, \ x_0 \in X \text{ given.} \qquad \text{(SP)}$$

- The corresponding functional equation (FE) has the form:

$$V(x) = \max_{x' \in \Gamma(x)} \left\{ \mathcal{F}(x, x') + \beta V(x') \right\}, \ \forall x \in X \qquad \text{(FE)}$$

General forms of SP and FE

Sometimes, you may see (SP) and (FE) written in other forms, e.g.,

$$W(x_0) = \max_{\{a_t\}_{t=0}^{\infty}} \left\{ \sum_{t=0}^{\infty} \beta^t \mathcal{F}(x_t, a_t) \right\} \quad \text{s.t. } a_t \in \Gamma(x_t),\ x_{t+1} = \phi(x_t, a_t),\ \forall t,\ x_0 \in X \text{ given.} \tag{SP}$$

$$V(x) = \max_{a \in \Gamma(x)} \left\{ \mathcal{F}(x, a) + \beta V(\phi(x, a)) \right\},\ \forall x \in X \tag{FE}$$

We stick to the forms on the previous page: SLP convention.

A few questions

Before we even try to solve the Bellman equation in (FE), we need to ask:

1. Does a solution V to the Bellman equation exist?
2. If it exists, is it unique?
3. If it exists and is unique, do we know how to find the solution V and the associated g?
4. If we find the unique solution V, do we know if $V(x_0)$ is the maximal value of (SP)?
5. Can we say something about the properties of V and g?

Fortunately, mild conditions lead to the first 4 "Yes"s, the properties of V largely follow those of $\mathcal{F}$, and g is usually a function. The properties of g are trickier and we may need to address it case by case.

A nice result: existence and uniqueness of $V = W$ and how to find it

Theorem 4
Assume X is convex, and Γ is nonempty, compact-valued, and continuous.
Assume $\mathcal{F}$ is bounded and continuous, and $\beta \in (0,1)$.
Let $C(X)$ be the space of bounded continuous functions $v : X \to \mathbb{R}$ with the sup norm.
Define the Bellman operator T on $C(X)$ by

$$(Tv)(x) = \max_{x' \in \Gamma(x)} \{\mathcal{F}(x, x') + \beta v(x')\}.$$

*Then T is a **contraction mapping** on $C(X)$. That is, $T : C(X) \to C(X)$ has a unique fixed point $v^* \in C(X)$, and for any $v_0 \in C(X)$,*

$$\|T^n v_0 - v^*\| \leq \beta^n \|v_0 - v^*\|, \quad \textit{i.e.,} \;\; T^n v_0 \to v^*.$$

And $v^*(x_0)$ is the maximal value in (SP).

Proof: SLP Thm 4.6 (contraction, unique fixed point, convergence rate); SLP Thm 4.3 for $v^*(x_0) = W(x_0)$ — its limit condition is automatic here as v is bounded.

A nice result: existence and uniqueness of $V = W$ and how to find it

Why is the theorem nice?

- In general, the two assumptions are relatively mild and easy to check. In the context of optimal growth, our earlier assumptions on u, F, and β are stronger than these two.
  - What about $\ln(\cdot)$ and other common utility functions being unbounded? In the context of optimal growth, we can restrict our attention to $X = [\epsilon, k_{\max}]$, where $\epsilon > 0$ is arbitrarily small, and $k_{\max} = \max\{\overline{k}_0, \overline{k}\}$ where $\overline{k}$ satisfies $f(\overline{k}) = \delta\overline{k}$. For more general cases, see SLP Ch.4.4.

Then, the rest of the theorem says...

- The solution V to the Bellman equation in (FE) exists and is unique
- We can find V using an iterative algorithm starting from an arbitrary bounded and continuous function on X (e.g., $v_0(x) = 0$, more on this later)
- $V(x_0) = W(x_0)$ in (SP)

More results on V's properties and g

The assumptions in the previous theorem hold. Then

- V **is strictly increasing** if $\mathcal{F}(x, x')$ is strictly increasing in x and Γ is monotone, i.e., $\Gamma(x_1) \subset \Gamma(x_2)$ if $x_1 < x_2 \in X$.

- V **is strictly concave and** g **is a single-valued function** if $\mathcal{F}$ is strictly concave and Γ is convex in the sense that $y_1 \in \Gamma(x_1)$ and $y_2 \in \Gamma(x_2)$ implies $\theta y_1 + (1-\theta) y_2 \in \Gamma(\theta x_1 + (1-\theta) x_2)$.

  Policy convergence Moreover, $g_n \to g$ pointwise if g_n is the policy function associated with $v_{n+1} = Tv_n$.

- V **is continuously differentiable** at $x \in X$ if $\mathcal{F}$ is strictly concave, Γ is convex, and $x \in intX$ and $g(x) \in int\Gamma(x)$. The derivatives satisfy

$$\frac{\partial V}{\partial x_i}(x) = \frac{\partial \mathcal{F}}{\partial x_i}(x, g(x)), \quad i = 1, \ldots, l. \qquad \textbf{(Envelope condition)}$$

  It is usually straightforward to show monotonicity of g as well.

The functional Euler equation

Back to our optimal growth setup

- It is easy to check that all assumptions required on the previous page hold. So the value function V as a solution to (FE) must then be strictly increasing, strictly concave, and continuously differentiable.
- The FOC of (FE) at the optimal $k' = g(k)$ can be written as:

$$u'(f(k) + (1-\delta)k - k') = \beta V'(k')$$

Remark: we can show $g \uparrow$ in k: LHS $\uparrow$ in k' and $\downarrow$ in k due to u's concavity, RHS $\downarrow$ in k' due to V's concavity, so when $k \uparrow$, LHS $\downarrow$, and k' must $\uparrow$ for RHS $\downarrow$.

- But what is $V'(k')$?

The functional Euler equation

- The envelope condition for the same problem at $k' = g(k)$ is

$$V'(k) = u'(f(k) + (1-\delta)k - k')[f'(k) + 1 - \delta]$$

for any interior k, so it must hold at $k' = g(k)$ as well, that is,

$$V'(k') = u'(f(k') + (1-\delta)k' - k'')[f'(k') + 1 - \delta]$$

where $k'' = g(k') = g(g(k))$ is the optimal capital stock in two periods

- Eliminating $V'(k')$ from the FOC yields the Euler equation again

$$\begin{aligned} u'(f(k) + (1-\delta)k - k') =& \beta V'(k') \\ =& \beta u'(f(k') + (1-\delta)k' - k'')[f'(k') + 1 - \delta] \end{aligned}$$

Looks familiar? Recall: backward induction in finite-horizon case.

How to find V and g (finally!)

- So far, we've addressed every question on our list on this page, except for a very practical problem: how to find V and g?
- In what follows, we show one analytical example where V and g can be solved by hand
- We don't have this luxury in general, so we'll introduce an algorithm called the **value function iteration** to solve for a Bellman equation

Roadmap: where we are

Theorem 4 did three jobs at once: it says V exists, is unique, and equals W. It also hands us an algorithm.

Economic question	**Mathematical object**	**Why *this* math**
How much should HH save?	max over infinite sequences (SP)	infinite horizon, dynamic decision
Intertemporal tradeoff	the **Euler equation**	gives economic intuition, not a solution
Limiting behavior at $t = \infty$	the **TVC**	replaces terminal condition
Same problem every period	the **Bellman equation** (FE)	*one function* instead of an infinite sequence
(SP) $\iff$ (FE)?	a few theorems	existence, uniqueness, *and* $V = W$
How do we solve it?	**value function iteration**	the algorithm implied by CMT

An analytical example

- Let $u(c) = \ln c$ and $f(k) = zk^\alpha$, $\alpha \in (0, 1)$. Assume full depreciation $\delta = 1$. Same as in the finite-horizon example.
- Bellman equation:

$$V(k) = (TV)(k) = \max_{k' \in [0, zk^\alpha]} \{\ln(zk^\alpha - k') + \beta V(k')\}$$

- First, we solve it using the method of undetermined coefficients (aka guess and verify)

An analytical example: method of undetermined coefficients

- Idea: guess a particular functional form of V (with coefficients TBD), then verify that TV has the exact same form and find out the coefficients
- Guess

$$V(k) = A + B \ln k$$

where A and B are the coefficients TBD

- Now find TV, i.e., solve the optimization problem on RHS

$$(TV)(k) = \max_{k' \in [0, zk^\alpha]} \{\ln(zk^\alpha - k') + \beta[A + B \ln(k')]\}$$

which leads to

$$\frac{1}{zk^\alpha - k'} = \frac{\beta B}{k'} \implies k' = g(k) = \frac{\beta B}{1 + \beta B} zk^\alpha \implies$$

$$(TV)(k) = \ln z + \alpha \ln k - \ln(1 + \beta B) + \beta \left[A + B \ln \frac{\beta B}{1 + \beta B} + B \ln z + B\alpha \ln k \right]$$

An analytical example: method of undetermined coefficients

- We see that $(TV)(k)$ indeed has a log-linear/affine form, so we solve for A and B by setting $(TV)(k) = A + B \ln k$:

$$\begin{aligned} A =& \ln z - \ln(1+\beta B) + \beta A + \beta B \ln \frac{\beta B}{1+\beta B} + \beta B \ln z \\ B =& \alpha + \alpha\beta B \end{aligned}$$

so

$$\begin{aligned} B =& \frac{\alpha}{1-\alpha\beta} \\ A =& \frac{1}{1-\beta}\left[\frac{\ln z}{1-\alpha\beta} + \ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta)\right] \end{aligned}$$

and, as expected, the policy function has the form

$$g(k) = \frac{\beta B}{1+\beta B} z k^{\alpha} = \alpha\beta z k^{\alpha}.$$

Algorithm: value function iteration

In general, we compute V exploiting the fact that T is a **contraction mapping** established in this theorem: starting from an arbitrary (bounded, continuous) function V_0:

$$V_n := T^n V_0 \to V$$

Algorithm:

1. Initialization: select any function $V_0(x)$, $\forall x$ (e.g., $V_0(x) = 0$);
2. Update: given $V_n(x)$ from the previous iteration, update V_{n+1} as $V_{n+1} = TV_n$, that is

$$V_{n+1}(x) = \max_{x' \in \Gamma(x)} \{\mathcal{F}(x, x') + \beta V_n(x')\}, \quad \forall x$$

3. Check convergence: if $\|V_{n+1} - V_n\| \leq \epsilon$, where ϵ is a pre-determined tolerance level, stop; otherwise, go back to step 2.

Algorithm: value function iteration

- What about the **policy function** g then?
- We use the policy convergence property when $\mathcal{F}$ is concave and Γ is convex
- That is, we obtain the associated policy $x' = g_{n+1}(x)$ when solving $V_{n+1} = TV_n$, then

$$g_n \to g$$

Looks familiar again? Recall: backward induction in the finite-horizon case. The interpretation of V_{n+1} is the maximal value in an economy with horizon n.

An analytical example: VFI

Same example: $u(c) = \ln c$, $f(k) = zk^\alpha$, $\delta = 1$. What if we solve it using VFI?

- Initialization: $V_0(k) = 0$, $\forall k \in [0, k_{\max}]$
- Iteration 1: $\forall k$,

$$\begin{aligned} V_1(k) =& (TV_0)(k) = \max_{k' \in [0, zk^\alpha]} \{\ln(zk^\alpha - k') + \beta \times 0\} \\ k' =& g_1(k) = 0, \\ V_1(k) =& \ln z + \alpha \ln k := A_1 + B_1 \ln k \end{aligned}$$

- Iteration 2: $\forall k$,

$$\begin{aligned} V_2(k) =& (TV_1)(k) = \max_{k' \in [0, zk^\alpha]} \{\ln(zk^\alpha - k') + \beta(\ln z + \alpha \ln k')\} \\ k' =& g_2(k) = \frac{\alpha\beta}{1 + \alpha\beta} zk^\alpha = S_2 zk^\alpha \\ V_2(k) =& (1 + \beta + \alpha\beta) \ln z - \ln(1 + \alpha\beta) + \alpha\beta \ln \frac{\alpha\beta}{1 + \alpha\beta} + (\alpha + \alpha^2\beta) \ln k := A_2 + B_2 \ln k \end{aligned}$$

Recall: 2-period example

An analytical example: VFI

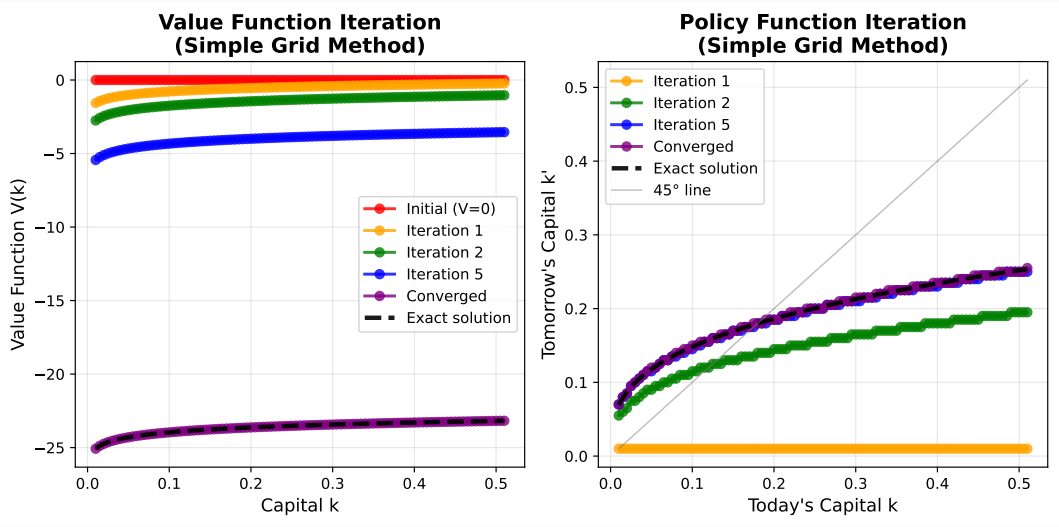
- Iteration n: $\forall k$,

$$\begin{aligned} V_n(k) =& (TV_{n-1})(k) = \max_{k' \in [0, zk^\alpha]} \{\ln(zk^\alpha - k') + \beta V_{n-1}(k')\} \\ k' =& g_n(k) = S_n z k^\alpha, \\ V_n(k) =& A_n + B_n \ln k \end{aligned}$$

$\vdots$

- Convergence: $V_n = T^n V_0 \to V$, i.e., $A_n \to A$ and $B_n \to B$ for some A, B

An analytical example: VFI



More about VFI

What we love about VFI?

- Guaranteed value convergence under mild conditions (contraction mapping theorem) Blackwell
- Simplest interpretation and implementation

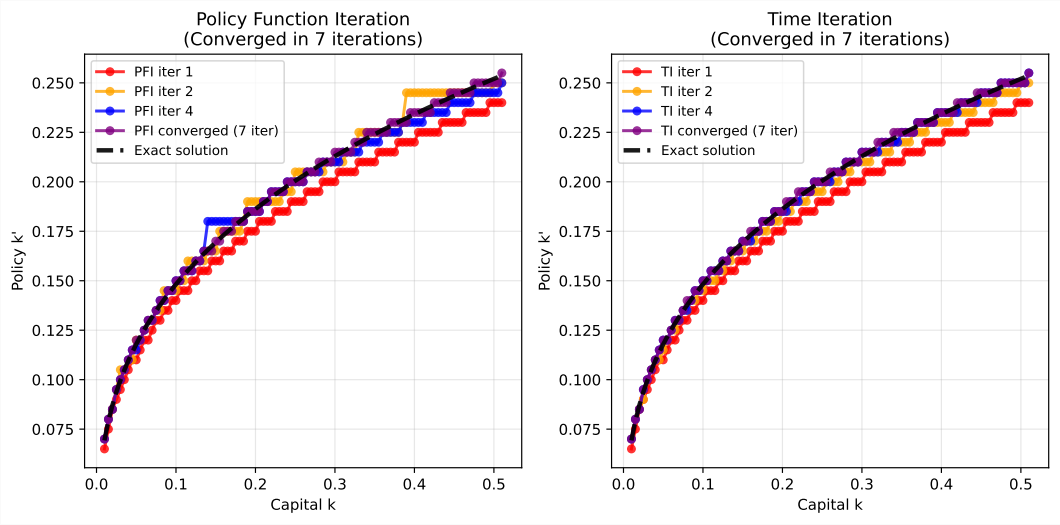
What we don't like about VFI?

- Slow and computationally costly if we do brute-force discretization

Other algorithms exist to improve computational efficiency

- In this course, you only need to know the basic VFI
- You'll learn about other algorithms (e.g., policy function iteration or time iteration) in 2nd year field courses

An analytical example: alternative algorithms



Roadmap: completed

Every row is now a thing we can do, and — more importantly — a thing we know *why* we do.

Economic question	**Mathematical object**	**Why *this* math**
How much should HH save?	max over infinite sequences (SP)	infinite horizon, dynamic decision
Intertemporal tradeoff	the **Euler equation**	gives economic intuition, not a solution
Limiting behavior at $t = \infty$	the **TVC**	replaces terminal condition
Same problem every period	the **Bellman equation** (FE)	*one function* instead of an infinite sequence
(SP) $\iff$ (FE)?	a few theorems	existence, uniqueness, *and* $V = W$
How do we solve it?	**value function iteration**	the algorithm implied by CMT

Hang on... where is the TVC in FE?

- Recall: we show that a sequence satisfying EE + TVC solves (SP)
- Recall: we derive the functional EE ($\Longleftarrow$ FOC+EC) from the Bellman eq'n (FE)
- Why didn't we discuss TVC in (FE)?
  - Short answer: TVC as a "selection rule" is already baked in the recursive form
  - For (SP) $\Longleftrightarrow$ (FE), we need $\lim_{t\to\infty} \beta^t V(k_t) = 0$ (SLP Thm 4.3), which serves as a selection rule ensured by $\beta \in (0,1)$ and bounded u
  - contraction $\Longrightarrow$ unique V and g (that *necessarily* satisfy the functional EE under differentiability)
- **Important**: it doesn't mean you can solve a Bellman equation simply by looking at the functional EE!

Steady state and the modified golden rule

Steady state: $k_{SS} > 0$ s.t. $g(k_{SS}) = k_{SS}$, i.e., if $k_0 = k_{SS}$, then $k^*_{t+1} = k_{SS}$, $\forall t$.

- How to find k_{SS}? The Euler equations:

$$u'(c) = \beta u'(c')[f'(k') + 1 - \delta]$$

At SS, $c = c' = c_{SS}$, $k = k' = k_{SS}$, so

$$f'(k_{SS}) = \frac{1}{\beta} + \delta - 1,$$

referred to as the "**modified golden rule**." It also implies $k_{SS} > 0$ is **unique**.

- Recall the original golden rule in the Solow model:

$$f'(k_{\text{Gold}}) = \delta < f'(k_{SS})$$

so $k_{SS} < k_{\text{Gold}}$.

Convergence to steady state

Theorem 5 (Monotonic convergence)
Given any $k_0 > 0$, let the sequence $\{k^*_{t+1}\}_{t=0}^{\infty}$ be generated by the policy function g s.t. $k^*_{t+1} = g(k^*_t)$. Then $k^*_{t+1} \to k_{SS}$ and the convergence is monotonic.

Proof.
Given the unique existence of $k_{SS} > 0$, it suffices to show that for any $k > 0$, $k < k_{SS}$ implies $k < k' = g(k) < k_{SS}$, and $k > k_{SS}$ implies $k > k' = g(k) > k_{SS}$.
Case $k < k_{SS}$: We established that g is strictly increasing, so $k' = g(k) < g(k_{SS}) = k_{SS}$.
To show $k' = g(k) > k$, we use the envelope condition and the Euler equation:

$$V'(k) = u'(c)[f'(k) + 1 - \delta], \qquad V'(k') = u'(c')[f'(k') + 1 - \delta] \overset{\text{Euler eqn}}{=} \frac{1}{\beta} u'(c).$$

The concavity of f implies that $f'(k) + 1 - \delta > f'(k_{SS}) + 1 - \delta = \frac{1}{\beta}$, so $V'(k) > V'(k')$.
Then, by the concavity of V established earlier, we have $k < k'$.
Case $k > k_{SS}$: Same logic. Omitted. □

A quick comparison with the Solow model

- Same monotonic convergence
- A big difference is that the endogenous saving rate here is usually a nontrivial function of capital stock instead of a constant:

$$s(k) = \frac{g(k) - (1-\delta)k}{f(k)}$$

Figure 1: Capital Stock Transition Dynamics

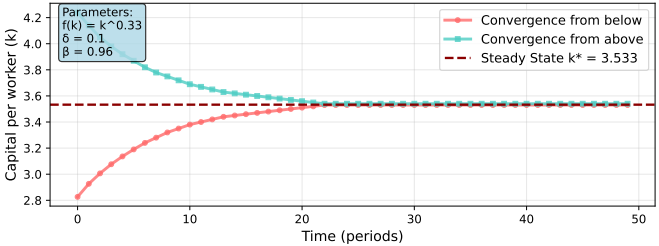
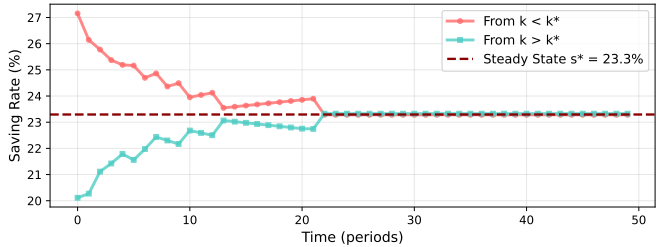


Figure 2: Saving Rate Transition Dynamics



With long-run growth

Source of growth

- So far, no source of growth, population or productivity
- Suppose we bring in population growth and technological advancement:

$$L_t = (1+n)^t, \quad A_t = (1+\gamma)^t$$

where we normalize $L_0 = 1$, $A_0 = 1$. Aggregate production still takes the form $F(K_t, A_t L_t)$.

- Aggregate resource constraint becomes

$$C_t + K_{t+1} \leq F(K_t, A_t L_t) + (1-\delta)K_t$$

or, after adjusting for efficiency unit of labor

$$\widetilde{c}_t + (1+n)(1+\gamma)\widetilde{k}_{t+1} \leq f(\widetilde{k}_t) + (1-\delta)\widetilde{k}_t$$

same as in the Solow model

Planner's objective

- We may formulate the planner's objective as to maximize individual lifetime utility

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

  or to maximize the combined lifetime utility of the whole economy

$$\sum_{t=0}^{\infty} \beta^t L_t u(c_t)$$

  we stick to the first choice.

- Written in growth-adjusted terms:

$$\sum_{t=0}^{\infty} \beta^t u\left((1+\gamma)^t \widetilde{c}_t\right)$$

CRRA utility

- We specify the per period utility function as a power function $u(c) = c^a$ with $a \leq 1$, more commonly written as

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}, \quad \sigma \geq 0$$

  known as the constant relative risk aversion (CRRA) utility function

- Up to a positive affine transformation, this is the only per-period utility function compatible with balanced growth

- Special cases: $\sigma = 0$ means linear utility or risk neutrality; $\sigma > 0$ means strictly concave utility function with $\sigma = 1$ representing $\ln c$ because

$$\lim_{\sigma \to 1} \frac{c^{1-\sigma} - 1}{1 - \sigma} \overset{\text{l'Hospital rule}}{=} \lim_{\sigma \to 1} \frac{-e^{(1-\sigma)\ln c} \ln c}{-1} = \ln c$$

SP and FE

- Planner's problem in sequential formulation or (SP) is

$$\max_{\{\widetilde{c}_t,\widetilde{k}_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \widetilde{\beta}^t \frac{\widetilde{c}_t^{1-\sigma}-1}{1-\sigma} + \text{const.}$$
$$\text{s.t. } \widetilde{c}_t + (1+n)(1+\gamma)\widetilde{k}_{t+1} \leq f(\widetilde{k}_t) + (1-\delta)\widetilde{k}_t, \forall t \ ,$$
$$\widetilde{c}_t, \widetilde{k}_{t+1} \geq 0, \quad \text{given } k_0.$$

  where $\widetilde{\beta} := \beta(1+\gamma)^{1-\sigma}$, and we **assume** $\widetilde{\beta} < 1$ so lifetime utility stays bounded — it can fail when σ is small and $\gamma > 0$

- Or, in recursive formulation, the Bellman equation

$$V(\widetilde{k}) = \max_{\widetilde{k}' \in [0, \frac{f(\widetilde{k})+(1-\delta)\widetilde{k}}{(1+n)(1+\gamma)}]} \left\{ \frac{[f(\widetilde{k}) + (1-\delta)\widetilde{k} - (1+n)(1+\gamma)\widetilde{k}']^{1-\sigma} - 1}{1-\sigma} + \widetilde{\beta} V(\widetilde{k}') \right\}$$

The Euler equation: similar to the no-growth case

- Same analysis as before:

$$\widetilde{c}^{-\sigma}(1+n)(1+\gamma) = \widetilde{\beta} V'(\widetilde{k}'), \tag{FOC}$$

$$V'(\widetilde{k}) = \widetilde{c}^{-\sigma}[f'(\widetilde{k}) + 1 - \delta] \tag{EC}$$

  where $\widetilde{c} = f(\widetilde{k}) + (1-\delta)\widetilde{k} - (1+n)(1+\gamma)\widetilde{k}'$.

- So the Euler equation becomes

$$\begin{aligned}\widetilde{c}^{-\sigma}(1+n)(1+\gamma) &= \widetilde{\beta}\widetilde{c}'^{-\sigma}[f'(\widetilde{k}') + 1 - \delta] \\ &= \beta(1+\gamma)^{1-\sigma}\widetilde{c}'^{-\sigma}[f'(\widetilde{k}') + 1 - \delta]\end{aligned}$$

Balanced growth path

- A balanced growth path means $\widetilde{k} = \widetilde{k}'$ and $\widetilde{c} = \widetilde{c}'$, so

$$f'(\widetilde{k}_{BGP}) = \frac{(1+n)(1+\gamma)}{\widetilde{\beta}} - 1 + \delta = \frac{(1+n)(1+\gamma)^{\sigma}}{\beta} - 1 + \delta$$

  and $\widetilde{c}_{BGP} = f(\widetilde{k}_{BGP}) - \underbrace{(n+\gamma+\delta+n\gamma)\widetilde{k}_{BGP}}_{=\widetilde{i}_{BGP}}$. This is the **modified golden rule with growth**: at $n = \gamma = 0$ it collapses to $f'(k_{SS}) = \frac{1}{\beta} - 1 + \delta$, still above Solow's $f'(k_{\text{Gold}}) = \delta$.

- The per capita and aggregate consumption and capital stock along the BGP:

$$\begin{aligned} c_{t,BGP} &= (1+\gamma)^t \widetilde{c}_{BGP}, & C_{t,BGP} &= [(1+n)(1+\gamma)]^t \widetilde{c}_{BGP}, \\ k_{t,BGP} &= (1+\gamma)^t \widetilde{k}_{BGP}, & K_{t,BGP} &= [(1+n)(1+\gamma)]^t \widetilde{k}_{BGP}, \end{aligned}$$

Taking stock

- What we did: we solve the planner's problem in the simplest setup of a neoclassical growth model with infinite horizon
  - Sequence problem (SP): we can discuss some "economics" using the Euler equations, but we typically cannot solve the problem
  - Functional equation (FE): a recursive formulation that's actually solvable
  - Under standard assumptions, to solve (FE) is to solve the original (SP)
- The hardest part is a change in your mindset to get used to (FE): think recursively!
- Most (?) macro papers with infinite horizon in discrete time ignore (SP) altogether and work exclusively with (FE), i.e., the Bellman equation

Cookbook procedures: a cheatsheet

Sequence problem (SP):

- Write down the Lagrangian
- Derive the FOCs or full KKT, then the EEs, interpret the economic intuitions
- Further analysis: analytically solve the problem, find a steady state, conduct comparative statics, etc.

Recursive formulation/Bellman equation/functional equation (FE):

- Translate (SP) into a Bellman equation
- Derive the FOCs, the ECs, then the functional EEs, interpret the economic intuitions
- Solve the problem, analytically or numerically
- Further analysis: find a steady state, conduct comparative statics, etc.

Readings

- AKMM: Ch.4
- Krueger: Ch.3.2, 4, 5
- SLP: Ch.2-4, 5.1

Blackwell's sufficient conditions for a contraction

back How do we know an operator is a contraction mapping in general?

Theorem 6 (Blackwell's sufficient conditions)
Let $X \subseteq \mathbb{R}^n$, and let $B(X)$ be the set of bounded functions $f : X \to \mathbb{R}$ with the sup norm. Let $T : B(X) \to B(X)$ be an operator satisfying

1. *(monotonicity) $f, g \in B(X)$ and $f \leq g$ implies $Tf \leq Tg$;*
2. *(discounting) $\exists \beta \in (0, 1)$ such that*

$$[T(f + a)](x) \leq (Tf)(x) + \beta a, \quad \forall f \in B(X),\ a \geq 0,\ x \in X,$$

where $(f + a)(x) = f(x) + a$.

Then T is a contraction mapping on $B(X)$ with modulus β.

Proof: See SLP Theorem 3.3.