ECON 720: Advanced Macroeconomic Theory I

The Solow Model

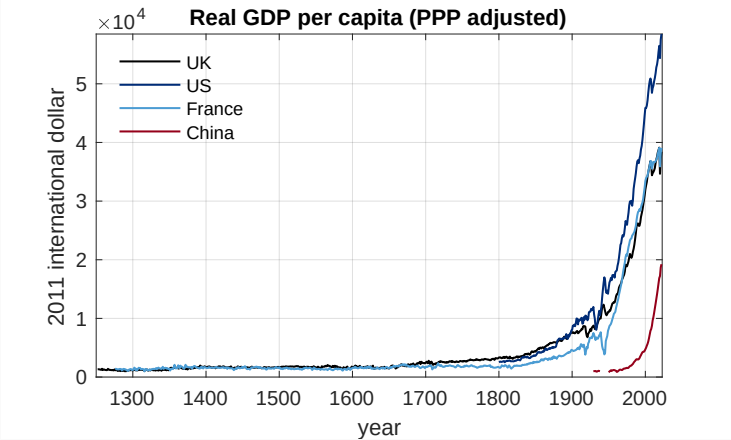
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Growth

Why no growth before 1800? Why sustained growth after 1800?



Data Source: Maddison Project Database

The Malthusian model

The Malthusian model



- Malthus (1798): An Essay on the Principle of Population
- Simple yet powerful model of the relation between population and economic growth
- Good description of human economies until $\sim$1800
- Idea: advances in food-producing technology $\rightarrow$ more food for a while $\rightarrow$ higher population $\rightarrow$ food per person reduced to old level
- "Malthusian trap," a no-growth prediction: no increase in the living standard in the long run, unless population growth is limited
- We translate Malthus's idea into a simple model

Model setup

- Discrete time, infinite horizon: $t = 0, 1, 2, \ldots$
- Constant-returns-to-scale (CRS) aggregate food production technology:

$$Y_t = zF(D, L_t)$$

  z is (Hicks neutral) productivity, D is **land** in fixed supply, L_t is time-t **labor** or population, Y_t is aggregate food output, F is twice differentiable, strictly increasing and concave in D and L, with $F(0, L) = 0$ and $\lim_{D\to\infty} F(D, L) = \infty$.

- Gross population growth rate is an increasing function of food per person $g\left(\frac{Y_t}{L_t}\right)$, so population evolves as

$$L_{t+1} = L_t g\left(\frac{Y_t}{L_t}\right) = L_t g\left(zF\left(\frac{D}{L_t}, 1\right)\right)$$

  assume: $g(0) = 0$, $g' > 0$, $g'' < 0$, $\lim_{c\to\infty} g(c) > 1$, $L_0 > 0$.

Steady state and convergence

Steady state: $L^* > 0$ s.t. $g\left(zF\left(\frac{D}{L^*}, 1\right)\right) = 1$, i.e., if $L_t = L^*$, then $L_{t+1} = L^*$.

Theorem 1
There exists a unique steady state $L^* > 0$ s.t. $\forall L_0 > 0$, $\lim_{t\to\infty} L_t = L^*$ and the convergence is monotone.

Proof.
Unique existence of $L^* > 0$ follows $F_1 > 0$, $g' > 0$, $g(0) = 0 < 1$ and $g(\infty) > 1$.

Monotone convergence means $L_t \uparrow L^*$ if $L_0 < L^*$ and $L_t \downarrow L^*$ if $L_0 > L^*$.

Case $L_0 < L^*$: we want to show that if $L_t < L^*$, then $L_t < L_{t+1} < L^*$. Observe that population growth $g(zF(D/L, 1))$ strictly decreases in L, so if $L_t < L^*$, $g(zF(D/L_t, 1)) > 1$ and $L_{t+1} > L_t$. Observe also that if $Lg(zF(D/L, 1))$ strictly increases in L, we have $L_{t+1} = L_t g(zF(D/L_t, 1)) < L^* g(zF(D/L^*, 1)) = L^*$. To show that $Lg(zF(D/L, 1))$ strictly increases in L is to show its first order derivative wrt L is positive, or $g(zF(D/L, 1)) - g'(zF(D/L, 1))zF_1(D/L, 1)D/L > 0$. To do so, let $h(x) = g(zF(x, 1))$, and it suffices to show $h(x) - xh'(x) > 0$ for $x > 0$. Note that $h(0) = 0$ because $g(0) = F(0, 1) = 0$ and that h is strictly increasing and concave because g and F are. By mean value theorem, $h(x) = xh'(\xi)$ for some $\xi \in (0, x)$, so $h(x) - xh'(x) > 0$ as h' strictly decreases. As such, we establish that $\{L_t\}_{t=0}^{\infty}$ is a bounded, strictly increasing sequence if $L_0 < L^*$, which has a limit. Approaching the limit means $L_{t+1}/L_t \to 1$, so this limit must be the unique L^* s.t. $g(zF(D/L^*, 1)) = 1$.

Case $L_0 > L^*$: similar and omitted. □

Implication

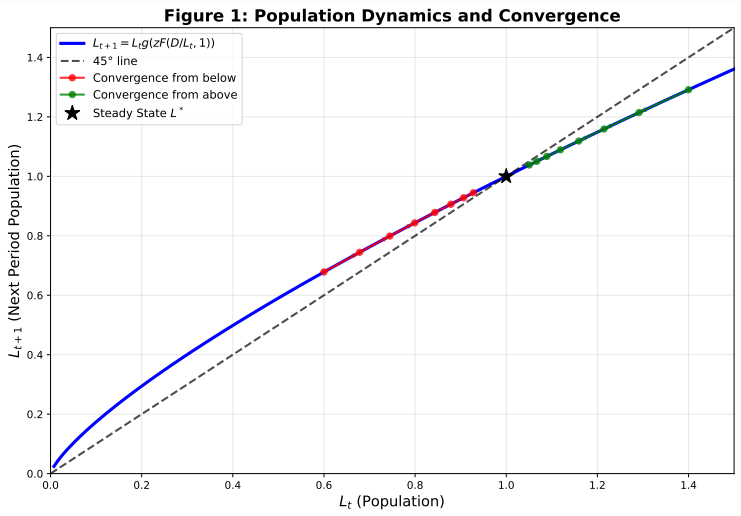
Malthusian trap: Define $c^* > 0$ such that $g(c^*) = 1$. Then c^* is the long-run living standard (food per person). Better food-production technology (z) or more land (D) increases population but does not alter c^* in the long run.

- Suppose we start from an initial steady state characterized by L_1^* associated with productivity level z_1
- Productivity jumps to $z_2 > z_1$ at time T and stays at z_2 hence forth
- Living standard jumps on impact to $c_T = z_2 F(D/L_1^*, 1) > c^*$
- Population grows from $T + 1$ onward
- However, population eventually approaches $L_2^* > L_1^*$ and living standard returns to c^*
- An increase in D leads to similar dynamics

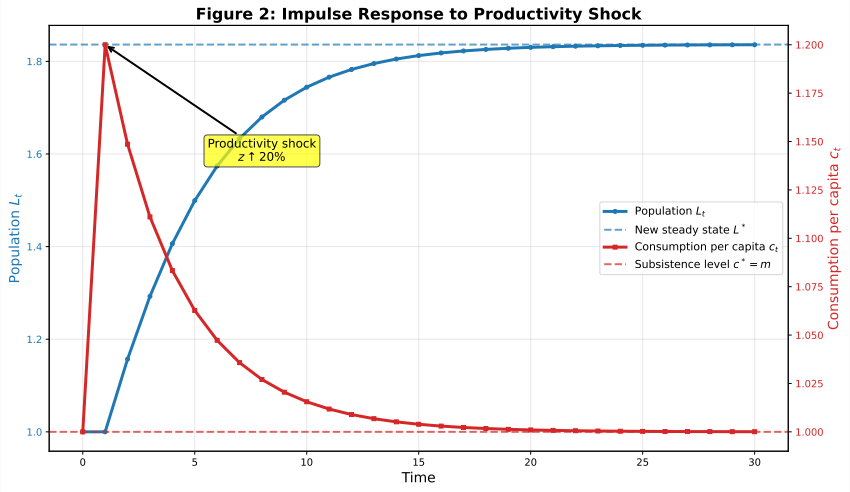
An example

- Consider specific functional forms: $F(D, L) = D^{\alpha} L^{1-\alpha}$ and $g(c) = (c/\boldsymbol{m})^{\gamma}$, where $\boldsymbol{m} > 0$ is a constant parameter and $\alpha, \gamma \in (0, 1)$
- Obviously, $c^* = \boldsymbol{m}$
- Population dynamics become $L_{t+1} = L_t g(zF(D/L_t, 1)) = (z/\boldsymbol{m})^{\gamma} D^{\alpha\gamma} L_t^{1-\alpha\gamma}$
- Living standard decreases in population $Y_t/L_t = z(D/L_t)^{\alpha}$
- Steady state: $L^* = (z/\boldsymbol{m})^{1/\alpha} D$

Convergence to the steady state



Transition path



Success and failure of the Malthusian model

What Malthusian model explains:

- Why most of human history shows no growth in living standard (Malthusian trap)
- Population adjusts to keep per capita income constant

But it cannot explain the sustained growth after 1800 (Industrial Revolution)

- Missing ingredient post Industrial Revolution? Solow (1956) said: capital accumulation.

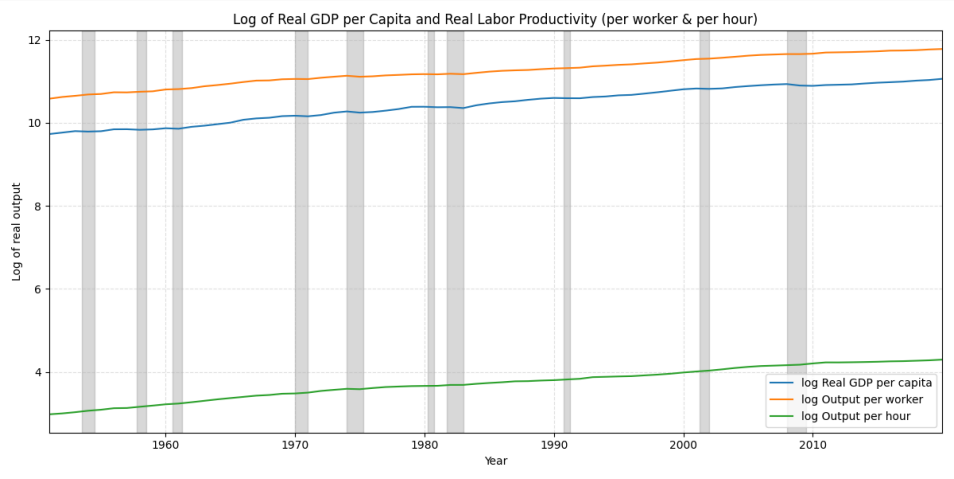
The Solow model

More growth facts (Kaldor's facts)

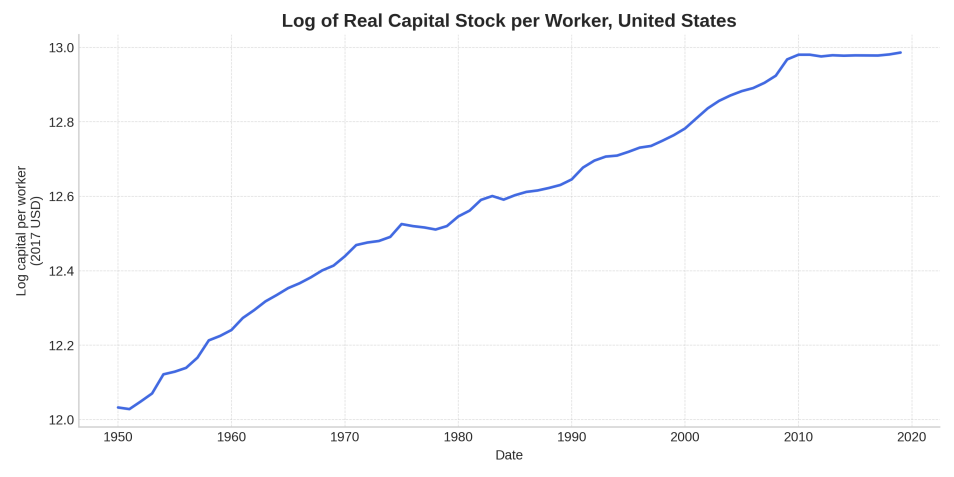
Stylized growth facts on most industrialized economies documented by Kaldor (1961)

1. Labor productivity has grown at a sustained rate.
2. Capital per worker has also grown at a sustained rate.
3. The real interest rate, or return on capital, has been stable.
4. The ratio of capital to output has also been stable.
5. Capital and labor have captured stable shares of national income.
6. There is sizable cross-country variation in growth rates.

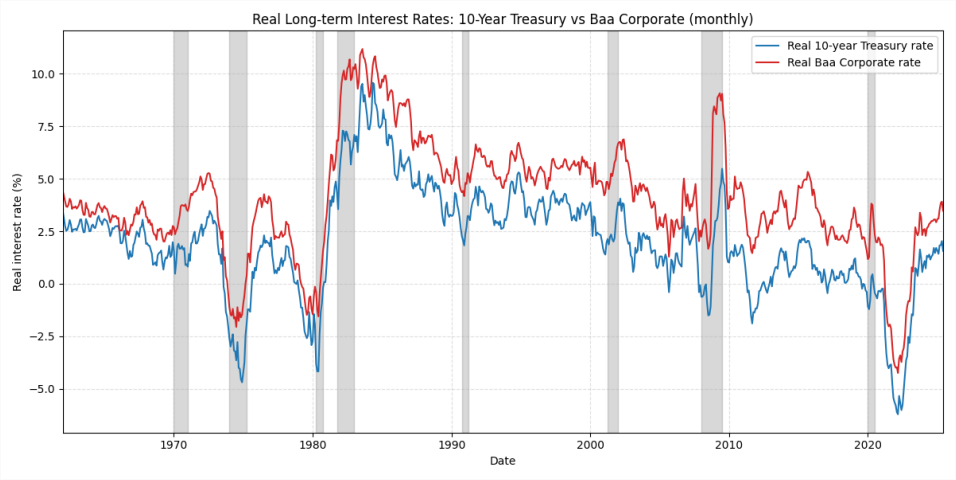
Fact 1. Labor productivity has grown at a sustained rate.



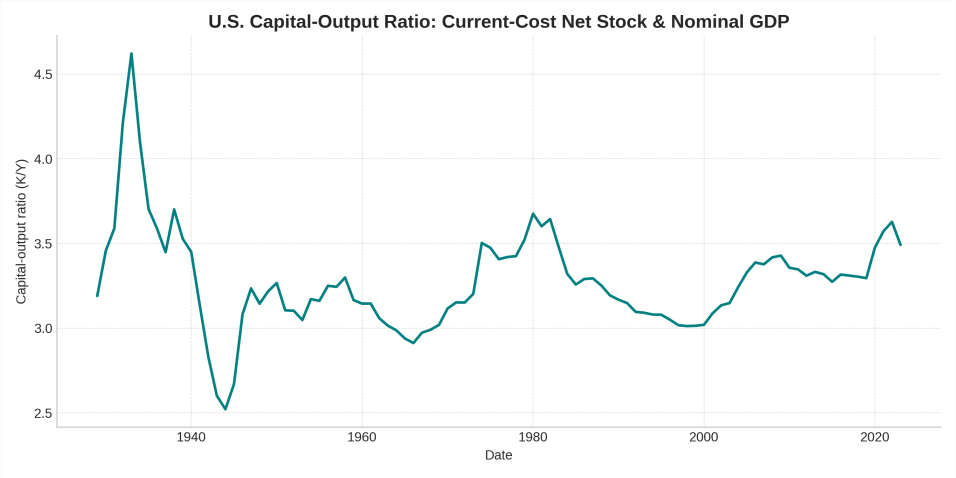
Fact 2. Capital per worker has also grown at a sustained rate.



Fact 3. The real interest rate, or return on capital, has been stable.

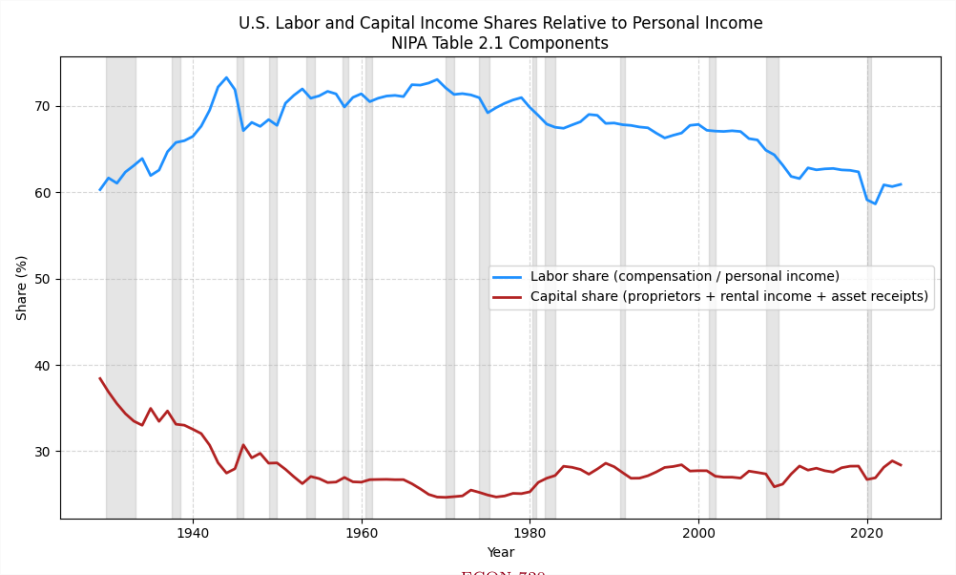


Fact 4. The ratio of capital to output has also been stable.

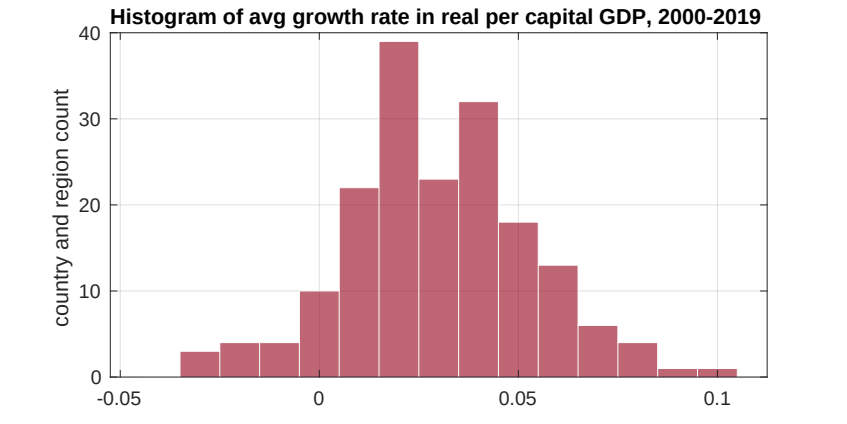


Caution: Capital is a stock concept and output is a flow concept. We need to specify the measurement frequency of output! Here, it's annual output.

Fact 5. Capital and labor have captured stable shares of national income.



Fact 6. There is sizable cross-country variation in growth rate.



The Solow model



- A simple model for the mechanics of economic growth in one economy/country and cross-country income differences
  - we focus on the one-economy case and omit the analysis on cross-country variation
- At the center of the Solow model
  - a neoclassical aggregate production function (neoclassical: diminishing marginal returns to each input)
  - capital accumulation dynamics (with an exogenous, fixed saving rate)
- We will state the model *as if* it is a true equilibrium model

A basic Solow model: setup

Simplest version: no population growth, technological progress, or human capital

- Discrete time, infinite horizon: $t = 0, 1, 2, \ldots$
- Agents: a fixed measure $L_t = L$ of households and a representative firm
- Goods/commodities traded in each period: labor service, capital service, a final output good (numeraire) that can be either consumed or invested
- Households work, rent out capital, consume, and make capital investment, $\forall t$
- The representative firm hires labor and rents capital, and uses them as inputs to produce the consumption good according to a given technology, $\forall t$
- All markets are perfectly competitive

A basic Solow model: setup

Households follow a *behavioral rule*, i.e., not the result of any optimization:

- each household is endowed with 1 unit of labor, supplied inelastically, so aggregate labor supply is L and aggregate labor income at t is $w_t L$ when market wage is w_t.
- households own aggregate capital K_t at t and supply it inelastically, and capital income at t is $r_t K_t$ when market interest rate is r_t; capital depreciates at rate $\delta \in (0, 1]$ per period; initial capital stock $K_0 > 0$ is exogenous.
- *behavioral rule for consumption and investment*: each household invests an exogenous, fixed fraction $s \in [0, 1]$ of income and consumes the rest, so:

$$
\begin{aligned}
C_t + I_t = & \underbrace{w_t L}_{\text{labor income}} + \underbrace{r_t K_t}_{\text{capital income}} \\
I_t = & s(w_t L + r_t K_t) \\
K_{t+1} = & (1 - \delta) K_t + I_t
\end{aligned}
$$

A basic Solow model: setup

The price-taking representative firm chooses capital and labor inputs to maximize profit in each period:

- production technology is represented by a time invariant *aggregate production function*

$$Y = F(K, L)$$

  with the following assumptions:

  - $F(K, L)$ exhibits CRS and is strictly increasing and concave in K and L
  - $F(0, L) = 0$, and (Inada conditions) $\lim_{K \to 0} F_1(K, L) = \infty$, $\lim_{K \to \infty} F_1(K, L) = 0$

- $\forall t$, given market prices w_t and r_t, the representative firm solves the following profit maximization problem

$$\max_{K,L>0} F(K, L) - r_t K - w_t L$$

Competitive equilibrium

Definition 2
Given $L_t = L > 0$ and $K_0 > 0$ and the households' behavioral rule, a **competitive equilibrium** of this economy consists of a sequence of prices $\{r_t, w_t\}_{t=0}^{\infty}$ and a sequence of allocation $\{C_t, I_t, K_t, K_t^d, L_t^d\}_{t=0}^{\infty}$ such that:

1. (household behavior) given $\{r_t, w_t\}_{t=0}^{\infty}$, $\{C_t, I_t, K_t\}_{t=0}^{\infty}$ describes household behavior;
2. (firm optimization) given $\{r_t, w_t\}_{t=0}^{\infty}$, $\{K_t^d, L_t^d\}_{t=0}^{\infty}$ solves the representative firm's profit-optimization problem;
3. (market clearing) all markets clear at all times

$$C_t + I_t = F(K_t^d, L_t^d), \quad K_t^d = K_t, \quad L_t^d = L, \quad \forall t.$$

Analysis

- We already know how households behave in this model. What about the firm?
- The firm maximizes per-period profit given prices. The FOCs imply

$$\begin{aligned} r_t =& F_1(K_t^d, L_t^d), \\ w_t =& F_2(K_t^d, L_t^d). \end{aligned}$$

  Recall that a CRS production function implies

$$F(K, L) = F_1(K, L) \cdot K + F_2(K, L) \cdot L, \quad \forall K, L > 0.$$

  It means that, when K_t^d and L_t^d are optimally chosen, we have $F(K_t^d, L_t^d) = r_t K_t^d + w_t L_t^d$, and the optimal profit is 0.

- Plugging in the market-clearing conditions yields $r_t = F_1(K_t, L)$ and $w_t = F_2(K_t, L)$, and

$$C_t + I_t = F(K_t, L) = r_t K_t + w_t L.$$

Analysis

- Note that our equilibrium *allocation* path can be written as

$$\begin{aligned}
&\text{(resource constraint)} && C_t + I_t = Y_t = F(K_t, L_t), \\
&\text{(capital accumulation)} && K_{t+1} = (1-\delta)K_t + I_t, \quad \text{with } K_0 > 0 \text{ given}, \\
&\text{(investment or saving rule)} && I_t = sF(K_t, L_t).
\end{aligned}$$

  This is the usual way to specify a basic Solow model in intermediate macro.

- The equilibrium path is fully characterized by the *fundamental equation* of the Solow model that describes the law of motion of capital stock, $\forall t$,

$$K_{t+1} = (1-\delta)K_t + sF(K_t, L).$$

  Whenever K_t is known, so is the rest of the equilibrium outcome.

Written in per capita terms

- Per capital variables:

$$c_t := \frac{C_t}{L}, \quad k_t := \frac{K_t}{L}, \quad i_t := \frac{I_t}{L}, \quad y_t := \frac{Y_t}{L}, \quad f(k_t) := \frac{F(K_t, L)}{L} = F(k_t, 1),$$

  where the last one is ensured by a CRS production function

- The fundamental equation for capital accumulation can be written in per capita terms, $\forall t$,

$$k_{t+1} = (1 - \delta)k_t + sf(k_t).$$

- Note that the equilibrium prices become

$$\begin{aligned} r_t =& F_1(K_t, L) = \frac{\partial(Lf(K_t/L))}{\partial K_t} = f'(k_t), \\ w_t =& F_2(K_t, L) = \frac{\partial(Lf(K_t/L))}{\partial L} = f(k_t) - k_t f'(k_t). \end{aligned}$$

Steady state

Steady state: $k^* > 0$ s.t. if $k_t = k^*$, then $k_{t+1} = k^*$. That is, $k^* > 0$ solves

$$\delta k^* = sf(k^*).$$

- The rest of the economy's steady-state outcome follows k^*, including factor prices

$$r^* = f'(k^*), \quad w^* = f(k^*) - k^* f'(k^*)$$

- The steady-state condition can be equivalently written as capital-output ratio:

$$\frac{k^*}{y^*} = \frac{k^*}{f(k^*)} = \frac{s}{\delta}$$

  Note that k^* (hence y^* and i^*) is a (strictly increasing) function of s. But not c^*! (Think when $s = 0$ and $s = 1$.)

- Note: aggregate variables C^*, I^*, K^*, Y^* are also time-invariant at the steady state

Convergence

Our equilibrium path is not necessarily a steady state unless $k_0 = k^*$. Good news is...

Theorem 3
There exists a unique steady state $k^* > 0$ *such that* $\forall k_0 > 0$, $\lim_{t\to\infty} k_t = k^*$ *and the convergence is monotone.*

The proof is similar to (and simpler than) that of the Malthusian model's steady state. Omitted.

- Interpretation: (**absolute convergence**) if two countries have identical $F(\cdot)$, s and δ, they eventually converge to the same steady state (per capita) regardless of their initial wealth k_0.

The Golden rule

- What is the "optimal" saving rate s? We don't know.
  - We can't do actual welfare analysis without properly defined household preferences/utility function!
- Rephrase the question: what is the saving rate that maximizes steady-state per capita consumption $c^* = (1-s)f(k^*)$?

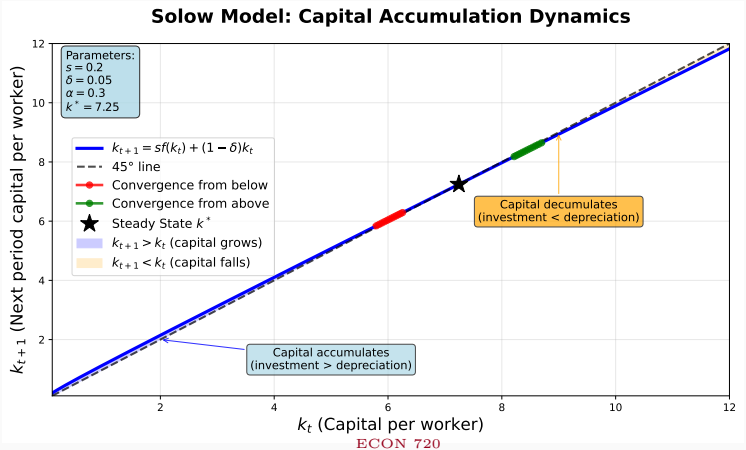
$$\max_{s\in[0,1]} c^*(s) = (1-s)f(k^*(s)) = f(k^*(s)) - \delta k^*(s),$$

  where c^* and k^* are written as functions of s.
- The resulting saving rate s_{Gold} is the "**golden rule**" saving rate, and the corresponding capital stock is $k^*_{\text{Gold}} = k^*(s_{\text{Gold}})$. FOC of the problem above implies

$$MPK_{\text{Gold}} = f'(k^*_{\text{Gold}}) = f'(k^*(s_{\text{Gold}})) = \delta.$$

An example

Consider the Cobb-Douglas aggregate production function $F(K, L) = K^{\alpha} L^{1-\alpha}$, that is, $f(k) = k^{\alpha}$. The capital and labor shares are always fixed at α and $1 - \alpha$ when firms optimize. Obviously, $k^* = (s/\delta)^{\frac{1}{1-\alpha}}$. It is straightforward to show that $s_{\text{Gold}} = \alpha$.



The Solow model with growth: setup

Now consider technological progress and population growth. Setup is the same except:

- The measure of households at t is L_t, growing at a constant rate $n > 0$ such that $L_{t+1} = (1+n)L_t$ with a given $L_0 > 0$. Household behavior is summarized by

$$\begin{aligned} C_t + I_t &= w_t L_t + r_t K_t \\ I_t &= s(w_t L_t + r_t K_t) \\ K_{t+1} &= (1-\delta)K_t + I_t \end{aligned}$$

- The representative firm's *aggregate production function* at time t becomes

$$Y = F(K, A_t L),$$

where *labor augmenting* (aka Harrod neutral) productivity A_t grows at a constant rate $\gamma > 0$ such that $A_{t+1} = (1+\gamma)A_t$ with a given $A_0 > 0$. $\forall t$, given prices w_t and r_t, firm problem becomes

$$\max_{K,L>0} F(K, A_t L) - r_t K - w_t L.$$

Competitive equilibrium

Definition 4
Given $L_0, K_0, A_0 > 0$, $L_{t+1} = (1+n)L_t$, $A_{t+1} = (1+\gamma)A_t$, and the households' behavioral rule, a **competitive equilibrium** of this economy consists of a sequence of prices $\{r_t, w_t\}_{t=0}^{\infty}$ and a sequence of allocation $\{C_t, I_t, K_t, K_t^d, L_t^d\}_{t=0}^{\infty}$ such that:

1. (household behavior) given $\{r_t, w_t\}_{t=0}^{\infty}$, $\{C_t, I_t, K_t\}_{t=0}^{\infty}$ describes household behavior;
2. (firm optimization) given $\{r_t, w_t\}_{t=0}^{\infty}$, $\{K_t^d, L_t^d\}_{t=0}^{\infty}$ solves the representative firm's profit-optimization problem;
3. (market clearing) all markets clear at all times

$$C_t + I_t = F(K_t^d, A_t L_t^d), \quad K_t^d = K_t, \quad L_t^d = L_t, \quad \forall t.$$

Analysis

Similar to the basic model:

- The firm maximizes per-period profit given prices. The FOCs imply

$$\begin{aligned} r_t =& F_1(K_t^d, A_t L_t^d), \\ w_t =& A_t F_2(K_t^d, A_t L_t^d). \end{aligned}$$

  And we still have

$$F(K_t^d, A_t L_t^d) = F_1(K_t^d, A_t L_t^d) \cdot K_t^d + F_2(K_t^d, A_t L_t^d) \cdot A_t L_t^d = r_t K_t^d + w_t L_t^d$$

  and the optimal profit is 0.

- Plugging in the market-clearing conditions yields $r_t = F_1(K_t, A_t L_t)$ and $w_t = A_t F_2(K_t, A_t L_t)$, and

$$C_t + I_t = F(K_t, A_t L_t) = r_t K_t + w_t L_t.$$

Analysis

- The equilibrium path is then fully characterized by a dynamic system of three variables

$$\begin{aligned} L_{t+1} =& (1+n)L_t, \\ A_{t+1} =& (1+\gamma)A_t, \\ K_{t+1} =& (1-\delta)K_t + sF(K_t, A_t L_t), \end{aligned}$$

  with $L_0, A_0, K_0 > 0$ given.

- Can we still get a steady state and the convergence result?
  - Yes, if we look at properly adjusted variables.
  - Aggregate and per capita variables will feature *balanced growth* (to be defined)

Written in efficient labor adjusted terms

- Define variables adjusted by *efficiency units of labor*, particularly $\widetilde{k}_t = \dfrac{K_t}{A_t L_t}$
- Divide both sides of aggregate capital accumulation rule by $A_t L_t$

$$\frac{K_{t+1}}{A_t L_t} = \frac{K_{t+1}}{A_{t+1} L_{t+1}} \frac{A_{t+1} L_{t+1}}{A_t L_t} = (1-\delta)\frac{K_t}{A_t L_t} + s\frac{F(K_t, A_t L_t)}{A_t L_t}$$

so we have

$$(1+\gamma)(1+n)\widetilde{k}_{t+1} = (1-\delta)\widetilde{k}_t + sf(\widetilde{k}_t)$$

- Factor prices can be written as

$$\begin{aligned} r_t =& F_1(K_t, A_t L_t) = \frac{\partial(A_t L_t f(K_t/(A_t L_t)))}{\partial K_t} = f'(\widetilde{k}_t) \\ w_t =& A_t F_2(K_t, A_t L_t) = A_t \frac{\partial(A_t L_t f(K_t/(A_t L_t)))}{\partial(A_t L_t)} = A_t[f(\widetilde{k}_t) - \widetilde{k}_t f'(\widetilde{k}_t)] \end{aligned}$$

Steady state and convergence for efficient labor adjusted variables

The **steady state** level of $\widetilde{k}^* > 0$ solves

$$[(1+\gamma)(1+n)+\delta-1]\widetilde{k}^* = sf(\widetilde{k}^*),$$

and it is straightforward to show that $\widetilde{k}^* > 0$ is still the convergence limit of $\{\widetilde{k}_t\}_{t=0}^{\infty}$ starting from any $\widetilde{k}_0 > 0$.

- The rest of the economy's outcome follows $\widetilde{k}^*$. Note that while real interest rate remains constant at the steady state

  $$r_t = r^* = f'(\widetilde{k}^*),$$

  real wage per unit of labor *grows at rate* γ,

  $$w_t = A_t[f(\widetilde{k}^*) - \widetilde{k}^* f'(\widetilde{k}^*)].$$

- What about aggregate and per capita allocation variables?

Balanced growth path for aggregate or per capita variables

Along a **balanced growth path** (BGP), all aggregate or per capita variables, i.e., (C_t, I_t, Y_t, K_t) or (c_t, i_t, y_t, k_t), grow at a common rate.

- An aggregate variable $X \in \{C, I, Y, K\}$ is simply $X_t = A_t L_t \widetilde{x}_t$, and the corresponding per capita variable is simply $x_t = A_t \widetilde{x}_t$.
- When $\widetilde{x}_t = \widetilde{x}^*$, it is easy to see that X_t has gross growth rate $(1+\gamma)(1+n)$, and x_t has gross growth rate $1+\gamma$: **a unique BGP**!

$$\frac{C_{t+1}}{C_t} = \frac{I_{t+1}}{I_t} = \frac{Y_{t+1}}{Y_t} = \frac{K_{t+1}}{K_t} = (1+\gamma)(1+n), \quad \frac{c_{t+1}}{c_t} = \frac{i_{t+1}}{i_t} = \frac{y_{t+1}}{y_t} = \frac{k_{t+1}}{k_t} = (1+\gamma)$$

- Because $\widetilde{k}_t \to \widetilde{k}^*$, $\forall \widetilde{k} > 0$, we know that this economy always converges to its unique BGP, $\forall K_0, A_0, L_0 > 0$.

Balanced growth path for aggregate or per capita variables

- Why did we assume a production function $F(K, AL)$ (instead of something like $AF(K, L)$)?
- Because it is the only CRS functional form consistent with a BGP.

Theorem 5 (Uzawa's theorem)
Suppose $Y_0 = F_0(K_0, L_0)$ s.t. F_0 is CRS. Suppose the economy is on a BGP s.t. $Y_t = (1+g)^t Y_0$, $K_t = (1+g)^t K_0$, and $L_t = (1+n)^t L_0$ for some $g, n > 0$. Then the sequence of production functions $\{F_t\}_{t=0}^{\infty}$ satisfying $Y_t = F_t(K_t, L_t)$ must have the form $F_t(K_t, L_t) = F_0(K_t, A_t L_t)$ for a sequence of $\{A_t\}_{t=0}^{\infty}$ s.t. $A_t = \left(\frac{1+g}{1+n}\right)^t$.

Proof.
F_0 is CRS, so we have $Y_t = (1+g)^t Y_0 = F_0((1+g)^t K_0, (1+g)^t L_0) = F_0\left(K_t, \left(\frac{1+g}{1+n}\right)^t L_t\right)$.
That is, $F_t(K_t, L_t) = F_0(K_t, A_t L_t)$ with $A_t = \left(\frac{1+g}{1+n}\right)^t$. □

Constant factor income shares

- Along the BGP, both the **capital income share** and the **labor income share** are constant, despite the growing real wage.
- To see why, recall that along any equilibrium path,

$$Y_t = F(K_t, A_t L_t) = \underbrace{r_t K_t}_{\text{capital income}} + \underbrace{w_t L_t}_{\text{labor income}},$$

  and firm makes zero profit.
- We only need to show that the capital income share remains constant along the BGP:

$$\frac{r^* K_t}{Y_t} = \frac{r^* K_t}{F(K_t, A_t L_t)} = \frac{r^* \widetilde{k}^*}{f(\widetilde{k}^*)},$$

  which is constant along the BGP.

The Solow model's predictions vs. Kaldor's facts

Along the BGP:

- labor productivity Y/L: typically defined as output per hour worked, same as output per capita/worker in this model. $Y_t/L_t = A_t\widetilde{y}^*$, growing at a constant (net) rate γ
- capital per worker K/L: $K_t/L_t = A_t\widetilde{k}^*$, growing at γ
- capital to output ratio K/Y: $K_t/Y_t = \widetilde{k}^*/f(\widetilde{k}^*)$, remaining constant, moreover

$$\frac{K_t}{Y_t} = \frac{\widetilde{k}^*}{f(\widetilde{k}^*)} = \frac{s}{(1+\gamma)(1+n)+\delta-1}$$

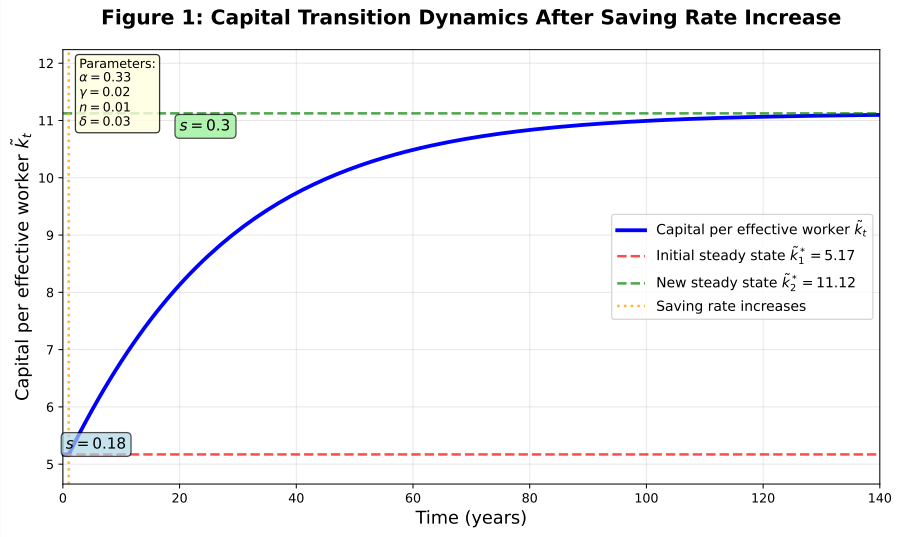
- real interest rate r: $r^* = f'(\widetilde{k}^*)$, remaining constant
- capital and labor shares rK/Y and wL/Y: constant
- cross-country variation in growth (omitted)

Calibration: assigning appropriate parameter values

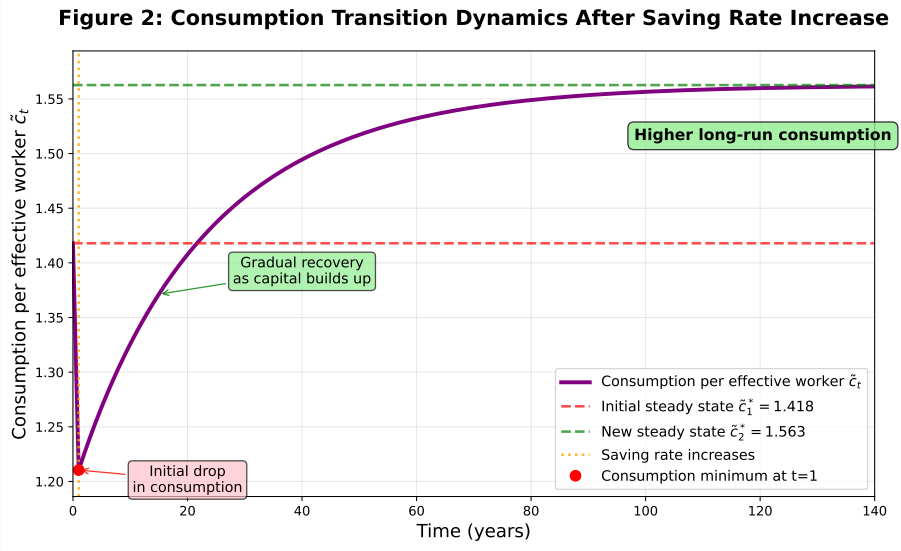
Assume a Cobb-Douglas production function $F(K, AL) = K^{\alpha}(AL)^{1-\alpha}$, $\alpha \in (0, 1)$. Assume period length is a year. Parameters to be calibrated are $(\alpha, s, \delta, \gamma, n, A_0, L_0)$. Calibrate to the US economy, assumed to be on its BGP:

- γ is growth rate of labor productivity or RGDP per capita, read from data: $\gamma = 0.02$
- n is population growth rate, read from data: $n = 0.01$
- α is the capital income share, i.e., $rK/Y = \alpha$ when K is optimally chosen, so $\alpha = 1/3$
- s is saving rate, matched to $s = I/Y$ or investment to GDP ratio, so $s = 0.18$
- δ is (annual) capital depreciation rate, we can borrow its value from the literature, or exploit K/Y: $3 = \frac{s}{(1+\gamma)(1+n)+\delta-1} \approx \frac{0.18}{0.02+0.01+\delta} \implies \delta = 0.03$
- A_0 and L_0 are irrelevant if we do not care about levels, so we normalize $A_0 = L_0 = 1$

An experiment: what if saving rate jumps to $s = 0.3$?



An experiment: what if saving rate jumps to $s = 0.3$?



Growth accounting

Another contribution of Solow is to introduce the growth accounting framework

- Aggregate output is produced by primary inputs given available technology
- So aggregate growth comes from changes in input usage and technology advancement
- How to (quantitatively) decompose aggregate growth into these components?
- Assumptions: A CRS aggregate production function using capital and labor as primary inputs; perfect competition.

Growth accounting

- To illustrate, suppose the aggregate production is

$$Y_t = z_t F(K_t, L_t),$$

  where z_t is TFP and $F(\cdot)$ exhibits CRS.

- Total differentiation

$$dY_t = F(K_t, L_t)dz_t + \underbrace{z_t F_1(K_t, L_t)}_{=r_t} dK_t + \underbrace{z_t F_2(K_t, L_t)}_{=w_t} dL_t$$

- Divide both sides by Y_t

$$\underbrace{\frac{dY_t}{Y_t}}_{\text{RGDP growth}} = \underbrace{\frac{dz_t}{z_t}}_{\text{TFP growth}} + \underbrace{\frac{r_t K_t}{Y_t}}_{\text{cap. share}} \times \underbrace{\frac{dK_t}{K_t}}_{\text{cap. growth}} + \underbrace{\frac{w_t L_t}{Y_t}}_{\text{labor share}} \times \underbrace{\frac{dL_t}{L_t}}_{\text{labor growth}}$$

Growth accounting

- Let capital share at t be α_t and labor share at t be $1-\alpha_t$. If we assume a Cobb-Douglas production function, then $\alpha_t = \alpha$ (constant over time).
- The equation above can be rewritten as

$$d \ln Y_t = d \ln z_t + \alpha_t d \ln K_t + (1-\alpha_t) d \ln L_t$$

  or, approximately in discrete time

$$\Delta \ln Y_t = \Delta \ln z_t + \alpha_t \Delta \ln K_t + (1-\alpha_t) \Delta \ln L_t$$

- TFP growth can thus be measured as the **Solow residual**

$$\Delta \ln z_t = \Delta \ln Y_t - \alpha_t \Delta \ln K_t - (1-\alpha_t) \Delta \ln L_t$$

  where everything on the right-hand side is observable from data.
- In practice, this simple approach has quite a few issues!

Concluding remarks on the Solow model

The Solow model is simple and powerful and serves as a great benchmark.

- It replicates stylized growth facts.
- Many ways to take it to data (we omitted several).

Where it falls short:

- It does not explain why productivity (the driving force of growth) keeps growing.
- Exogenous and fixed saving rate: it can't address policy or welfare related issues.

What's next: the neoclassical growth framework

- Proper microfoundations endogenizing household decisions
- Policy and welfare analysis

Readings

- AKMM Ch.2, Ch.3